THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and forty-fifth regular meeting of the Society was held in New York City on Saturday, October 30, 1909, extending through a morning and an afternoon session. About forty persons were in attendance, including the following twenty-seven members of the Society:

Professor G. D. Birkhoff, Professor E. W. Brown, Professor A. S. Chessin, Professor F. N. Cole, Miss L. D. Cummings, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor C. C. Grove, Professor C. N. Haskins, Professor W. H. Jackson, Mr. S. A. Joffe, Dr. L. C. Karpinski, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. C. Krathwohl, Professor W. W. Landis, Professor J. H. Maclagan-Wedderburn, Mr. H. H. Mitchell, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Dr. W. M. Strong, Professor Oswald Veblen, Mr. C. B. Walsh, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams.

Vice-President Edward Kasner occupied the chair at the morning session, being relieved at the afternoon session by Ex-Presidents Osgood and White. The Council announced the election of the following persons to membership in the Society: Dr. H. T. Burgess, University of Wisconsin; Professor H. H. Dalaker, University of Minnesota; Mr. G. C. Evans, Harvard University; Mr. Louis Gottschall, New York City; Dr. J. V. McKelvey, Cornell University; Miss H. H. MacGregor, Yankton College; Mr. H. H. Mitchell, Princeton University; Mr. U. G. Mitchell, Princeton University; Mr. R. R. Shumway, University of Minnesota; Dr. H. L. Slobin, University of Minnesota; Mr. I. W. Smith, University of North Dakota. Four applications for admission to membership in the Society were received.

The list of nominations for officers and other members of the Council to be placed on the official ballot for the annual meeting was reported. Mr. C. B. Upton was appointed Assistant Librarian of the Society.

In memory of Ex-President Simon Newcomb the following resolutions, presented by the Council, were adopted by the Society:

Resolved: That the American Mathematical Society record its great regret at the loss to the Society and to Science occasioned by the death of Professor Simon Newcomb.

As a member of the Society from the time when its scope became national in character and as one of its early Presidents, he showed his interest in its development. His advice and influence greatly contributed to the success which has attended its efforts toward the organization and progress of mathematical science throughout the American continent.

His achievements in many branches of thought and particularly in astronomical science will entitle him to high rank amongst those who have labored to advance knowledge and to place it within the reach of every student.

Resolved: That these resolutions be recorded in the minutes of the Society and that a copy of them be sent by the Secretary to the family of the late Professor Newcomb.

The following papers were read at this meeting:

(1) Professor C. N. HASKINS: "Note on the extremes of functions."

(2) Professor P. A. LAMBERT: "On the solution of linear differential equations."

(3) Professor FLORIAN CAJORI: "A note on the history of the slide rule."

(4) Professor CARL RUNGE: "A hydrodynamic problem treated graphically."

(5) Professor EDWARD KASNER: "The motion of particles starting from rest."

(6) Professor G. A. MILLER: "Note on the groups generated by two operators whose squares are invariant."

(7) Professor C. N. MOORE: "On the uniform convergence of the developments in Bessel functions of order zero."

Professor Runge was introduced by the Secretary. In the absence of the authors, the papers of Professor Lambert, Professor Cajori, Professor Miller, and Professor Moore were read by title.

Professor Miller's paper appears in full in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Haskins discusses a representation of the maximum maximorum and minimum minimorum of a function of a real variable as the limit of a definite integral. If in the interval $0 \leqq x \leqq 1$, $f(x)$ is continuous and m and M respectively its greatest lower and least upper bounds, and

$$J(z) = \tfrac{1}{2} \log \int_0^1 e^{zf(x)}\, dx,$$

then

$$\lim_{z=+\infty} J(z) = M, \quad \lim_{z=-\infty} J(z) = m,$$

and

$$\lim_{z=0} J(z) = \int_0^1 f(x)dx.$$

Moreover $J(z)$ is a monotonic increasing function of z, a fact which is proved by repeated use of the integral inequality of Schwarz. The theorems are closely related to those recently published by Dunkel,* but the proofs proceed by methods of the integral calculus.

2. The object of Professor Lambert's paper is to present a method of building up the solution of ordinary differential equations with algebraic coefficients. The method consists of the following steps: (*a*) Break up the given differential equation

$$(1) \qquad f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$$

into two parts, one of which equated to zero gives a differential equation which is readily solved by the usual methods, and introduce a parameter t as a factor of the other part, so that the given equation becomes

$$(2) \quad f_1\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) + f_2\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) t = 0.$$

(*b*) Assume that

$$(3) \qquad y = A + Bt + Ct^2 + Dt^3 + \cdots + Nt^n + \cdots,$$

where $A, B, C, D, \cdots, N, \cdots$ are undetermined functions of x, makes equation (2) an identity. Determine $A, B, C, D, \cdots, N, \cdots$ by solving the differential equations found by equating

* *Annals of Mathematics*, sec. 2, vol. 11 (1909), p. 26.

to zero the coefficients of the successive powers of t in this identity.

(c) Substitute these values of $A, B, C, D, \cdots, N, \cdots$ in (3) and place t equal to unity; then

$$(4) \qquad y = A + B + C + D + \cdots + N + \cdots,$$

provided this series terminates or is convergent, is a solution of the differential equation.

3. It is shown by Professor Cajori that the earliest German description of the rectilinear slide rule, found in Leupold's Theatrum arithmetico-geometricum, Leipzig, 1727, and copied by Leupold from an old manuscript of hitherto unknown authorship, is a translation of passages in Seth Partridge's Description and use of an instrument called the double scale of proportion, which appeared at London in several editions. The earliest edition bears the date 1662, and not 1671, as hitherto supposed.

4. The amount of water flowing from a reservoir of infinite depth over a weir of given section may be approximately found by graphical methods, while there do not exist any methods of treating the problem analytically. In Professor Runge's paper the viscosity is neglected and the problem is considered in two dimensions only. The velocity potential is deduced by starting with a plausible assumption of the free surface. On this curve the velocity of the water, i. e., the gradient of the velocity potential, is known. From these data the stream lines and lines of constant potential may be constructed and any stream line may serve as a possible section of a weir. For practical purposes it is only necessary to approximate the given weir at the top, because the rest of it will have little influence on the amount of flow.

5. Professor Kasner showed that when a particle, starting from rest, is acted on by any positional field of force, it describes a path whose curvature is one third the curvature of the line of force through the given point. The case where the curvature is zero is studied separately, a simple result being obtained concerning the lowest non-vanishing derivatives. The discussion is extended to constrained motion and to systems of particles.

7. In Professor Moore's paper the following theorem is established: If in the interval $0 \leqq x \leqq 1$ the function $f(x)$ is continuous, save at a finite number of points at which it has a finite jump, and if in each interval of continuity it has a second derivative that is finite and integrable, then the development of $f(x)$ in Bessel functions of order zero will converge uniformly to the value $f(x)$ throughout the interval $0 \leqq x \leqq x_0 < c$, where c is the smallest value of x for which $f(x)$ is discontinuous.

It is a well known fact that, under much wider conditions than those imposed here, the development will converge uniformly to $f(x)$ throughout any closed interval that does not include nor reach up to a point of discontinuity of $f(x)$, and does not reach up to the origin. The uniform convergence of the series in the neighborhood of the origin, however, does not seem to have been discussed.

F. N. COLE,
Secretary.

NOTE ON THE GROUPS GENERATED BY TWO OPERATORS WHOSE SQUARES ARE INVARIANT.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, October 30, 1909.)

LET s_1, s_2 be any two operators which satisfy the conditions

$$s_1^{-1} s_2^2 s_1 = s_2^2, \quad s_2^{-1} s_1^2 s_2 = s_1^2.$$

From these equations it results directly that both s_1^2 and s_2^2 are invariant under the non-abelian group G generated by s_1, s_2. On the other hand the equations

$$s_1^{-1} \cdot s_1 s_2^{-1} \cdot s_1 = s_2^{-1} s_1 = (s_1 s_2^{-1})^{-1} \cdot s_1^2 s_2^{-2} = s_2^{-1} \cdot s_1 s_2^{-1} \cdot s_2,$$

$$s_2^{-1} \cdot s_2 s_1^{-1} \cdot s_2 = s_1^{-1} s_2 = (s_2 s_1^{-1})^{-1} \cdot s_1^{-2} s_2^2 = s_1^{-1} \cdot s_2 s_1^{-1} \cdot s_1$$

show that the abelian group generated by the three operators $s_1^2, s_2^2, s_1 s_2^{-1}$ is invariant under G, and also that the quotient group of G with respect to the group H generated by s_1^2, s_2^2 is dihedral. Hence it results also that H is the central of G whenever $s_1 s_2^{-1}$ is of odd order. When $s_1 s_2^{-1}$ is of even order, the central of G is either H or it includes H as a subgroup of half its order. Since the quotient group of G with respect to an abelian invariant subgroup is dihedral G must be solvable.

The object of the present note is to show clearly that G may be regarded as a generalization of the dihedral group which includes the earlier generalization obtained by considering the groups generated by two operators which have a common square.* If two operators have a common square, this square is clearly invariant under the group generated by these operators; but if the squares are invariant under this group they evidently are not necessarily the same. From this it follows directly that the present generalization includes the earlier one, and it gives rise to an almost equally elementary category of groups as a result of the equations established in the preceding paragraph. If two operators have a common square, it is known that the product of either one into the inverse of the other is transformed into its inverse by each of the operators. The analogous theorem as regards the operators under consideration may be expressed as follows:

When each of two operators is commutative with the square of the other, the product of one into the inverse of the other is transformed by each of the two operators into its inverse multiplied by an invariant operator under the group generated by the two operators.

UNIVERSITY OF ILLINOIS.

THE SOLUTION OF THE EQUATION IN TWO REAL VARIABLES AT A POINT WHERE BOTH THE PARTIAL DERIVATIVES VANISH.

BY DR. L. S. DEDERICK.

(Read before the American Mathematical Society, September 14, 1909.)

IF $F(x, y)$ is a real function of the real variables x and y which is continuous at and near the point (x_0, y_0), and vanishes at this point, but has one first order partial derivative at the point not equal to zero, there are a number of well-known theorems about the existence of other values of x and y satisfying the equation

$$F(x, y) = 0, \tag{1}$$

* *Archiv der Mathematik und Physik*, vol. 9 (1905), p. 6.

and about the properties of y as a function of x, or x as a function of y, thus defined. The case, however, where both the first partial derivatives of $F(x, y)$ vanish at (x_0, y_0), seems to have received very little attention, although analogous theorems for this case are tacitly used in most treatments of the singularities of plane curves. In this subject, of course, it is frequently assumed that $F(x, y)$ is analytic at the point in question; but even where much less than this is demanded, it frequently happens that more is required than is at all necessary,* that the exact conditions are not stated, and that existence proofs are omitted.

The object of the present paper is to examine the possible cases that may arise when the first partial derivatives both vanish, and to state theorems analogous to Dini's theorem in these cases. No attempt has been made to reduce the sufficient conditions to an absolute minimum, but in most cases it seems unlikely that any simple statement could be made of conditions materially less restrictive. The general statement may be made concerning the requirements of continuity, that there must be one order of partial derivatives of $F(x, y)$ continuous at and near (x_0, y_0) and not all vanishing at that point, and derivatives of any higher orders must be continuous at and near (x_0, y_0) whose values at (x_0, y_0) are required in order to determine the character of the solution or the value of a derivative of y with respect to x whose existence is asserted; i. e., we may perform the formal differentiations for ascertaining any desired facts, and know that they are justified if all the partial derivatives of $F(x, y)$ that we use are continuous at and near (x_0, y_0).

In all the following theorems we shall assume that $F(x, y)$, together with all its partial derivatives of order n or lower, is continuous at and near (x_0, y_0), and that $F(x, y)$ and all its derivatives of order less than n, vanish at this point, but that not all the derivatives of order n are equal to zero there. The following notation will be used throughout:

$$h = x - x_0, \quad k = y - y_0, \quad \alpha = \frac{k}{h} = \frac{y - y_0}{x - x_0},$$

$$F_{ij}(x, y) = \frac{\partial^{i+j} F(x, y)}{\partial x^i \partial y^j}, \quad F_{ij} = F_{ij}(x_0, y_0),$$

* E. g., de la Vallée-Poussin, Cours d'analyse, vol. 2, chap. 8.

$$b_{ij} = \frac{1}{i!\,j!}, \quad bF_{ij} = b_{ij}F_{ij},$$

$$\overline{F}_{ij} = F_{ij}(x_0 + \theta h,\ y_0 + \theta_1 k).$$

where $0 \leqq \theta \leqq 1$, $0 \leqq \theta_1 \leqq 1$. In most cases $0 < \theta = \theta_1 < 1$, but not always. Moreover θ or θ_1 need not preserve the same value throughout any one discussion.

THEOREM 1. — *There is a neighborhood of* (x_0, y_0) *in which all pairs of values of* x *and* y *satisfying equation* (1) *can be grouped into not more than* n *branches such that on each branch* α *approaches a limit (finite or infinite) equal to some real root of the* n*th degree equation in* α

$$\sum_{i=0}^{n} bF_{n-i,\,i}\alpha^i = 0. \tag{2}$$

By Taylor's theorem for two variables there is a neighborhood of (x_0, y_0) in which we can write

$$F(x, y) = h^n \sum_{i=0}^{n} b\overline{F}_{n-i,\,i}\alpha^i. \tag{3}$$

If this vanishes when $x \neq x_0$, then

$$\sum_{i=0}^{n} b\overline{F}_{n-i,\,i}\alpha^i = 0. \tag{4}$$

From the continuity of the partial derivatives each coefficient in (4) approaches the corresponding coefficient in (2) as (x, y) approaches (x_0, y_0), and hence each root of (4) approaches some root of (2). There is, therefore, a neighborhood of (x_0, y_0) in which the possible solutions of (1) are separated into not more than n branches according to the root of (2) whose neighborhood contains the corresponding value of α. If $F_{0,n} = 0$, equation (2) must be regarded, of course, as having an infinite root; and accordingly, there may be a branch on which α becomes infinite. This includes the case where $F(x_0, y) = 0$ throughout some interval including y_0. Here (3) is satisfied by the vanishing of h. In the subsequent theorems, however, we shall always understand by a root of equation (2) a real

finite root. The question of a solution belonging to an infinite root is easily reduced to this by interchanging the variables x and y.

By a *branch* here is meant merely a certain aggregate of pairs of values. We have not shown that a branch defines y as a single valued function of x, nor yet a continuous function. We have shown, however, that the values of x and y belonging to one branch satisfy a relation

$$y - y_0 = (x - x_0)(\alpha_1 + \zeta) = h(\alpha_1 + \zeta),$$

where α_1 is a root of (2) and ζ approaches zero with h, and that in a sufficiently small neighborhood of (x_0, y_0) any solution belongs to some one such branch.*

THEOREM 2. — *Corresponding to any simple root α_1 of equation* (2), *there is a neighborhood D of (x_0, y_0) and a single-valued function $f(x)$ such that when $y = f(x)$ in D, $F(x, y) \equiv 0$. This function is continuous and has a continuous derivative in D, and represents all the values of x and y in D which satisfy* (1) *and belong to the branch corresponding to α_1;*

i. e., there is an interval $(\alpha_1 - \gamma, \alpha_1 + \gamma)$, such that if (x, y) is in D, and $\alpha_1 - \gamma < \alpha < \alpha_1 + \gamma$, then $F(x, y) = 0$ if and only if $y = f(x)$.

If (x, y) and (x_1, y_1) are any two points in the neighborhood of (x_0, y_0),

$$F(x, y) = F(x_1, y_1) + [F(x, y) - F(x_1, y_1)].$$

If $x = x_1$,

$$F(x, y) - F(x_1, y_1) = F(x, y) - F(x, y_1) = \Delta y F_{01}(x, y_1 + \theta \Delta y),$$

* NOTE. — For this theorem the conditions of continuity may be slightly reduced. It is sufficient to demand that the partial derivatives of $F(x, y)$ of order n shall exist at and near (x_0, y_0) and be continuous at that point. We can then stop the Taylor's development one term sooner and expand each partial derivative of order $(n-1)$ by the unsymmetrical form of the law of the mean,

$$F_{n-1-i,\,i}(x_0 + \theta h, y_0 + \theta k) = \theta h F_{n-i,\,i}(x_0 + \theta_1 \theta h, y_0) + \theta k F_{n-i-1,\,i+1}(x_0 + \theta h, y_0 + \theta_2 \theta k).$$

This gives the same result as the Taylor's development except for the factor $n\theta$, which does not affect the reasoning.

where $\Delta y = y - y_1$. In general,

$$y = y_0 + \alpha(x - x_0).$$

In particular, let

$$y_1 = y_0 + \alpha_1(x - x_0).$$

Then

$$\Delta y = y - y_1 = (\alpha - \alpha_1)(x - x_0) = h\Delta\alpha,$$

where $\Delta\alpha = \alpha - \alpha_1$. Hence

$$F(x, y) = F(x, y_1) + h\Delta\alpha F_{01}(x, y_1 + \theta\Delta y). \tag{5}$$

Now by Taylor's theorem,

$$F_{01}(x, y) = h^{n-1}\sum_{i=0}^{n-1} b_{n-1-i,\,i}\overline{F}_{n-1-i,\,i+1}\alpha^i = h^{n-1}\sum_{i=1}^{n} ib\overline{F}_{n-i,\,i}\alpha^{i-1}. \tag{6}$$

As h approaches zero and α approaches α_1, the coefficient of h^{n-1} in this expression approaches

$$\sum_{i=1}^{n} ibF_{n-i,\,i}\alpha_1^{i-1} = M,$$

where $M \neq 0$ since α_1 is a simple root of (2). If then we take any positive ϵ such that $\epsilon < |M|$, we can find a neighborhood D_1 of (x_0, y_0) and an interval $(\alpha_1 - \gamma, \alpha_1 + \gamma)$, such that when (x, y) is in D_1, and $\alpha_1 - \gamma < \alpha < \alpha_1 + \gamma$, we have

$$\left|\sum_{i=1}^{n} ibF_{n-i,\,i}(x, y)\alpha^{i-1}\right| > \epsilon. \tag{7}$$

Again, from (3),

$$F(x, y_1) = h^n\sum_{i=0}^{n} b\overline{F}_{n-i,\,i}\alpha_1^i. \tag{8}$$

Since α_1 is a root of (2), there is a neighborhood D_2 of (x_0, y_0) in which

$$\left|\sum_{i=0}^{n} bF_{n-i,\,i}(x, y)\alpha_1^i\right| < \epsilon|\Delta\alpha|. \tag{9}$$

Substituting in (5) from (8) and (6), we have

$$F(x, y) = h^n\left[\sum_{i=0}^{n} b\overline{F}_{n-i,\,i}\alpha_1^i + \Delta\alpha\sum_{i=1}^{n} ib\overline{F}_{n-i,\,i}\alpha^{i-1}\right].$$

From (7) and (9), if (x, y) lies in D, the region common to D_1 and D_2, and $\alpha_1 - \gamma < \alpha < \alpha_1 + \gamma$, then the first term in the bracket is less in absolute value that $\epsilon|\Delta\alpha|$, and the second is greater. If then we leave h unchanged but replace $\Delta\alpha$ by its negative, we change the sign of $F(x, y)$. Therefore $F(x, y)$ must vanish between $y = y_1 - \Delta y$ and $y = y_1 + \Delta y$; and since in this interval $F_{01}(x, y)$ does not vanish, it remains of one sign, and hence $F(x, y)$ can not vanish more than once. Thus for any value of x in D, there is one and only one value of y between $y = y_0 + h(\alpha_1 - \gamma)$ and $y = y_0 + h(\alpha_1 + \gamma)$ for which $F(x, y) = 0$. This defines y as a single-valued function of x in D, and makes it the complete solution which satisfies the condition $\lim_{x=x_0}\alpha = \alpha_1$. This last condition shows that $f(x)$ is continuous at $x = x_0$, and that $f'(x_0) = \alpha_1$. When $y = f(x)$ in D and $x \neq x_0$, we have seen that $F_{01}(x, y) \neq 0$. Hence by Dini's theorem, $f(x)$ and $f'(x)$ are continuous in D when $x \neq x_0$.

It only remains now to show that $f'(x)$ is continuous at $x = x_0$. When $x \neq x_0$, we know that

$$f'(x) = -\frac{F_{10}(x, y)}{F_{01}(x, y)} = -\frac{h^{n-1}\sum_{i=0}^{n-1} b_{n-1-i,\,i}\overline{F}_{n-i,\,i}\alpha^i}{h^{n-1}\sum_{i=0}^{n-1} b_{n-1-i,\,i}\overline{F}_{n-1-i,\,i+1}\alpha^i}$$

$$= -\frac{\sum_{i=0}^{n-1}(n-i)b\overline{F}_{n-i,\,i}\alpha^i}{\sum_{i=1}^{n} ib\overline{F}_{n-i,\,i}\alpha^{i-1}}.$$

As x approaches x_0 and hence α approaches α_1, the numerator and denominator of this both approach limits, and we have seen that the limit of the latter is not zero. Therefore $\lim_{x=x_0} f'(x)$ exists. But since $f(x)$ is continuous at x_0, this is sufficient that $f'(x)$ be continuous there also.*

This same general method of proof may be extended to the case where α_1 is a multiple root. The method is essentially this: to determine an approximate solution and a strip includ-

* Dini, Funzioni di variabili reali, § 75; or E. W. Hobson, Theory of functions of a real variable, § 220.

ing it within which the actual solution must lie, and then to show that on the approximation $F(x, y)$ is so small, and throughout the strip $F_{01}(x, y)$ is so large, that for any value of x, $F(x, y)$ must vanish for one and only one value of y in the strip. In Theorem 2, the approximation and the boundaries of the strip are straight lines. For later cases the approximation must be closer and the boundaries of the strip must approach each other more rapidly.

When $n = 1$, this theorem reduces to Dini's theorem and the proof becomes that given by de la Vallée-Poussin (Cours d'analyse infinitésimale, I, ¶ 142). As in the case of Dini's theorem, we can show here that every additional order of partial derivatives of $F(x, y)$ which are continuous at and near (x_0, y_0) insures the existence and continuity of an additional derivative of $f(x)$. This is shown in the following theorem, which completes the treatment of a simple root of equation (2).

THEOREM 3. — *If $f(x)$ is a solution of the sort discussed in Theorem* 2, *and the partial derivatives of $F(x, y)$ of order $(n + m)$ are continuous at and near (x_0, y_0), then $f^{(m+1)}(x)$ exists and is continuous at and near x_0.*

We know from Theorem 2 that this is true when $m = 0$. To show that it is true in general, we assume that $f^{(m)}(x)$ exists and is continuous at and near x_0. This we know is true when $m = 0$ or $m = 1$. It is also true if this theorem holds for all values of m less than the particular one in question. The proof, then, on the assumption that $f^{(m)}(x)$ is continuous, establishes the theorem by complete induction.

We can expand $F(x, y)$ by Taylor's theorem in powers of h from h^n to h^{n+m} with coefficients that are polynomials in α and contain no other variable except in the case of the last. From the assumption of the continuity of $f^{(m)}(x)$, we can write when $y = f(x)$

$$\alpha = f' + \tfrac{1}{2}hf'' + \cdots + \frac{1}{m!}h^{m-1}(f^{(m)} + \zeta), \tag{10}$$

where ζ approaches zero with h. If we substitute this in the expansion of $F(x, y)$ and arrange according to powers of h, the coefficients of the powers up to and including h^{n+m-2} will be constants. That of h^{n+m-1} will contain ζ linearly but no other variable. Hence, since $F(x, y) = 0$ when $y = f(x)$, we have an equation of the form,

$$A_n h^n + \cdots + A_{n+m-2}h^{n+m-2} + (B + \omega\zeta)h^{n+m-1} + h^{n+m}\phi(h) = 0,$$

where A_n, $\cdots$, A_{n+m-2}, and B are are constants, ω and ϕ approach limits as h approaches zero, and

$$\lim_{h=0} \omega = \frac{1}{m!} \sum_{i=1}^{n} ibF_{n-i,\,i}\alpha_1^i \neq 0.$$

Since ζ approaches zero with h,

$$A_n = A_{n+1} = \cdots = A_{n+m-2} = B = 0$$

and

$$\omega\zeta + h\phi(h) = 0.$$

Therefore $\zeta = h\eta$, where $\lim_{h=0} \eta$ exists. If we put this in (10), we have a development of α including the power h^m, in which the coefficient of h^{m-1} is constant. From the continuity of $f^{(m)}$, moreover, we have

$$y' = f'(x) = f' + hf'' + \cdots + \frac{h^{m-1}}{(m-1)!}(f^{(m)} + \zeta_1),$$

where ζ_1 approaches zero with h. When $y = f(x)$ we have also

$$F_{10}(x, y) + y'F_{01}(x, y) = 0.$$

If we expand the partial derivatives by Taylor's theorem and replace α and y' by their developments, we find that the powers of h in the expansions of the partial derivatives run from h^{n-1} to h^{n+m-1}, and hence after the substitution, the coefficients are constants up to and including that of h^{n+m-3}, while the coefficient of h^{n+m-2} involves ζ_1 linearly and no other variable. In the same way as for ζ, we show that $\zeta_1 = h\eta_1$ where $\lim_{h=0} \eta_1$ exists. This extends the development of y' one step further. In like manner, by substituting in the successive total derivatives of $F(x, y)$ with regard to x and equating them to zero, we add one power of h to the known developments of each of the derivatives, y'', y''', $\cdots$, $y^{(m)}$, finally getting $y^{(m)} = f^{(m)} + h\eta_m$, where $\lim_{h=0} \eta_m$ exists. By Dini's theorem we know that when $x \neq x_0$, $y^{(m+1)}$ exists, and hence the $(m+1)$th total derivative of $F(x, y)$ exists and is equal to zero. If we expand this in powers of h and substitute for α, y', $\cdots$, $y^{(m)}$ their developments in powers of h, we have an equation for determining $y^{(m+1)}$, in which the coefficient of $y^{(m+1)}$ is $F_{01}(x, y)$, or

$$h^{n-1} \sum_{i=1}^{n} ib\overline{F}_{n-i,\,i}\alpha^i.$$

In the rest of the equation the coefficients are constants up to and including that of h^{n-2}; and the coefficient of h^{n-1} approaches a limit. We cannot infer here as before that the constant coefficients are all zero, because we do not know that $y^{(m+1)}$ approaches a limit as h approaches zero. We can infer, however, that when h approaches zero, $y^{(m+1)}$ either approaches a limit or else becomes infinite. But since $f^{(m+1)}(x)$ exists when $x \neq x_0$, we know that $y^{(m)} = f^{(m)} + hf^{(m+1)}(x_0 + \theta h)$; and we have seen that $y^{(m)} = f^{(m)} + h\eta_m$. Therefore $f^{(m+1)}(x_0 + \theta h) = \eta_m$, and this approaches a limit. But this is impossible if $f^{(m+1)}$ becomes infinite. Therefore $\lim_{x=x_0} f^{(m+1)}(x)$ exists and from this it follows that $f^{(m+1)}(x)$ is continuous at $x = x_0$.

There now remains to consider the case of a multiple root of equation (2). We know from the theory of plane curves that in this case the existence and character of the solution depend upon the values of derivatives of an order higher than n. The simplest case of this sort is treated in the following theorem.

THEOREM 4. — *Let $F(x, y)$ have all its partial derivatives of order $(n+1)$ continuous at and near (x_0, y_0), and let α_1 be a multiple root of order m of equation (2); but let*

$$\sum_{i=0}^{n+1} bF_{n+1-i,i}\alpha_1^i \neq 0.$$

Then the values of x and y in the neighborhood of (x_0, y_0) for which $F(x, y) = 0$ and α approaches α_1 are completely given by a solution in the form $x = x_0 + et^m$, $y = y_0 + e\alpha_1 t^m + t^{m+1} f(t)$, where e is $+1$ or -1 and $f(t)$ is a single-valued function which is continuous at $t = 0$.

We shall omit the proof of this and proceed at once to a much more general discussion, from which this theorem follows as a special case. Let us assume that the partial derivatives of $F(x, y)$ of order $n + r$ are continuous at and near (x_0, y_0). Then we can expand $F(x, y)$ in the form,

$$F(x, y) = h^n \phi_n + h^{n+1}\phi_{n+1} + \cdots + h^{n+r}\bar{\phi}_{n+r}, \tag{11}$$

where ϕ_i is a polynomial of degree i in α, having for coefficients the partial derivatives of $F(x, y)$ of order i with the arguments (x_0, y_0) except where the dash as in $\bar{\phi}_{n+r}$ indicates the arguments $x_0 + \theta h, y_0 + \theta k$. Since the coefficients of $\bar{\phi}_{n+r}$ are continuous at and near (x_0, y_0), we may write $\bar{\phi}_{n+r} = \phi_{n+r} + \zeta$, where ζ approaches zero with h. Then at any point of a solution

$$(12) \qquad \phi_n + \phi_{n+1}h + \phi_{n+2}h^2 + \cdots + \phi_{n+r}h^r + \zeta h^r = 0.$$

We have seen in Theorem 1 that for any solution, α must approach a root of ϕ_n, which we have called α_1. In Theorems 2 and 3 we have discussed the case where α_1 is a simple root. Theorem 4 supposes that α_1 is a multiple root of ϕ_n but not a root of ϕ_{n+1}. We shall now allow α_1 to be a root of any number of ϕ's and of any possible orders, and obtain certain necessary conditions for a solution.

Let α_1 be a root of each of the polynomials, $\phi_n, \phi_{n+1}, \cdots, \phi_{n+r}$; and let its orders of multiplicity as a root of these be respectively $m_0, m_1, \cdots, m_r$. The important case that α_1 is not a root of ϕ_{n+r} is included here by making $m_r = 0$. We may assume without loss of generality that no previous m is zero, as that case is treated by taking a smaller value of r. For any solution belonging to α_1, we have $\alpha = \alpha_1 + \zeta_1$ where ζ_1 approaches zero with h. If we put this value of α in (12) and expand, the terms in ζ_1 of degree less than m_i in the expansion of ϕ_i will drop out from the vanishing of their coefficients, and we shall have

$$(13) \qquad \psi_0(\zeta_1)\zeta_1^{m_0} + \psi_1(\zeta_1)h\zeta_1^{m_1} + \cdots + \psi_r(\zeta_1)h^r\zeta_1^{m_r} + \zeta h^r = 0,$$

where the ψ's are polynomials in ζ_1 with constant coefficients, and the constant term in each is not zero. In fact

$$\psi_i(0) = \frac{1}{m_i!}\left(\frac{\partial}{\partial\alpha}\right)^{m_i}\phi(\alpha_1) \neq 0.$$

Now let c and d be any two integers that are relatively prime, and let e be $+1$ or -1, its sign being at present undetermined. It is always possible, however, to choose its sign so that for any value of h there is at least one value of t such that $h = et^d$. If we define t in this way and set $\eta = \zeta_1/t^c$, we have $\zeta_1 = \eta t^c$. Replacing h and ζ_1 in (13) by these values, we get

$$(14) \qquad \sum_{i=0}^{r} e^i\psi_i(\eta t^c)\eta^{m_i}t^{cm_i+di} + \zeta e^r t^{dr} = 0.$$

From this we may cancel out the lowest power of t that appears and get an algebraic equation that η must satisfy. If now we let t approach zero and impose the condition that η shall remain finite, we see that η must approach a root of the limiting form of equation (14). This is

$$\sum_i e^{ti}\psi_i(0)\eta^{m_i} = 0, \tag{15}$$

the sum including only such terms as had the minimum exponent of t in (14). In general this equation will consist of only one term, and hence its only solution is $\eta = 0$. If now we wish to exclude the case where η approaches zero with t, we must choose c and d so that two or more exponents of t in (14) are equal to each other and less than any others. Let $\lambda = c/d$. Then the exponent $cm_i + di$ becomes $d(\lambda m_i + i)$. As d is independent of i, the exponent is a minimum when $\lambda m_i + i$ is a minimum. For small values of λ, $\lambda m_0 < 1$, and hence the exponent of the first term is less than any other;

$$\lambda m_0 < i + \lambda m_i \qquad (i = 1, 2, \cdots, r).$$

Let λ_1 be the smallest value of λ for which this is not true, i. e., for which there is an i such that

$$\lambda m_0 = i + \lambda m_i.$$

Let this value of i be i_1, or if it happens simultaneously for two or more values of i, let i_1 be the greatest of these. Then it is easy to show that for increasing values of λ, the exponent $d(i_1 + \lambda m_{i_1})$ becomes and remains the minimum until it becomes equal to the exponent of some later term,

$$i_1 + \lambda m_{i_1} = i_2 + \lambda m_{i_2} \leqq i + \lambda m_i,$$

where $i_2 > i_1$. If there are two or more possible values of i_2, then i_2 is to indicate the largest of these. In like manner $d(i_2 + \lambda m_{i_2})$ becomes and remains the minimum exponent until it becomes equal to some exponent where $i > i_2$.

We proceed in this way, determining all possible values of λ (not more than r in number) for which η does not become zero or infinite as t approaches zero, but subject always to the condition that $i + \lambda m_i \leqq r$, as otherwise the term $\zeta e^{t}t^{ir}$ may become the principal infinitesimal, since we have no means of comparing the infinitesimals ζ and t. We can take account of this condition by setting m_r arbitrarily equal to zero. If $m_r = 0$, then $r + \lambda m_r (= r)$ appears in the equation for determining the last value of λ, and this value is to be treated like any previous one. If $m_r \neq 0$, then the coefficient of η^{m_r} in (15) is zero and $\eta = 0$ is a root. In this case there is a possible value of η

which approaches zero for all values of λ for which we can determine its behavior with the assumptions that we have made. We cannot say whether or not it yields a solution unless we know the existence and values of derivatives of order greater than $(n + r)$.

If η corresponds to an actual solution of (1), the identity of this solution is not affected by varying λ. Let us consider, then, the case that η belonging to this solution becomes infinite for all values of λ. This means in particular that the coefficient of the highest power of η in (14) approaches zero with h when λ has the values which make the exponent λm_0 the minimum. This, however, shows that the coefficient of the highest power of η involves a ψ which is not ψ_0. If then we set $r = 0$, this power of η will not appear. But we know from Theorem 1 that setting $r = 0$ gives the necessary condition for all solutions which approach (x_0, y_0). As we are concerned only with these, the case that η becomes infinite for all values of λ is disposed of. If η approaches zero for some values of λ and becomes infinite for others, it is easy to show that there is an intermediate value of λ for which it approaches a limit not zero.

Thus for any solution approaching (x_0, y_0) whose character is determined by the assumed data, we have a certain value of λ and an equation of the form (15) for determining the limiting value of η, or to write the equation more explicitly,

$$(16) \qquad e^p\psi_p(0)\eta^{m_p} + e^q\psi_q(0)\eta^{m_q} + \cdots + e^s\psi_s(0)\eta^{m_s} = 0,$$

where

$$(17) \quad cm_p + dp = cm_q + dq = \cdots = cm_s + ds = \mu \leqq cm_i + di.$$

Or, since $\eta \neq 0$,

$$(18) \quad e^{p-s}\psi_p(0)\eta^{m_p-m_s} + e^{q-s}\psi_q(0)\eta^{m_q-m_s} + \cdots + \psi_s(0) = 0.$$

We now have that any possible solution is expressible in the form

$$x = x_0 + et^d,$$

$$y = y_0 + et^d(\alpha_1 + \eta t^c) = y_0 + a_1t^d + a_2t^{d+c} + \zeta_2t^{d+c},$$

where

$$a_1 = e\alpha_1, \quad a_2 = \lim_{t=0} e\eta, \quad \text{and} \quad \lim_{t=0} \zeta_2 = 0;$$

and where c, d, e, and the limiting value of η, satisfy (17) and (18), and $e = \pm 1$.

These are the necessary conditions for a solution. If, however, the limit of η is a real simple root of (18), they are also sufficient. For let

$$y_1 = y_0 + a_1 t^d + a_2 t^{d+e}.$$

Then

$$F(x, y_1) = h^n \left(\sum_{i=0}^{r} e^i \psi_i(\eta t^e) \eta^{m_i} t^{em_i + di} + \zeta e^r t^{dr} \right)$$

$$= t^{nd+\mu} \omega(t), \quad \text{where } \lim_{t=0} \omega(t) = 0.$$

Developing $F_{01}(x, y)$ in the same form as (6), we get

$$F_{01}(x, y) = h^{n-1} \left(\sum_{i=0}^{r} h^i \frac{\partial \phi_{n+i}}{\partial \alpha} + \zeta' h^r \right),$$

where ζ' approaches zero with h. Substituting $\alpha = \alpha_1 + \zeta_1$ in $\partial \phi_{n+i} / \partial \alpha$, we get

$$F_{01}(x, y) = h^{n-1} \left(\sum_{i=0}^{r} h^i \zeta_1^{m_i - 1} m_i \psi_i(\zeta_1) + \zeta' h^r \right)$$

$$= e^{n-1} t^{(n-1)d} \left(\sum_{i=0}^{r} e^i m_i \psi_i(\zeta_1) \eta^{m_i - 1} t^{id + (m_i - 1)e} + \zeta' e^r t^{rd} \right)$$

$$= t^{(n-1)d+\mu-e} \Omega(t, \eta),$$

where the limit of $\Omega(t, \eta)$ is the partial derivative of the expression in (16) and hence not equal to zero on the assumption that η approaches a simple root. Since Ω is a continuous function of t and η, we can find small intervals for t and η within which $|\Omega| > \epsilon$, where ϵ is some positive constant. If now

$$y = y_0 + a_1 t^d + e\eta t^{d+e},$$

η having any fixed value in its interval, then

$$\Delta y = y - y_1 = (e\eta - a_2) t^{d+e} = e\Delta\eta \cdot t^{d+e}.$$

$$F(x, y) - F(x, y_1) = \Delta y F_{01}(x, y_1 + \theta \Delta y) = e\Delta\eta \cdot t^{nd+\mu} \Omega(t, ea_2 + \theta\Delta\eta).$$

$$F(x, y) = t^{nd+\mu} [\omega(t) + e\Delta\eta\Omega(t, ea_2 + \theta\Delta\eta)].$$

As ω approaches zero and Ω approaches some other limit, we can take t so small that

$$|\omega| < |\Omega \cdot \Delta\eta|,$$

and hence the sign of $F(x, y)$ depends on the sign of $\Delta\eta$. As in Theorem 2, it now follows that $F(x, y)$ vanishes once and only once in the strip and hence there is a single valued function of t which yields the complete solution belonging to this limiting value of η. A short computation will show that if we change the sign of e and make no other change, we either get the same solution that we had before or else none at all.

All this is on the assumption that η approaches a simple root of (18). If, however, η approaches a multiple root, we substitute $\eta = e(a_2 + \zeta_2)$ in (14) and arrange according to powers of t, up to t^{dr}. We then have an equation connecting ζ_2 and t of exactly the same form as equation (13) connecting ζ_1 and h. With this we can proceed in the same way, getting ζ_2 and t expressed in powers of a new parameter as a necessary condition, which becomes sufficient if the equation for the new coefficient has a simple root. For a multiple root we must repeat the process. With regard to the continued repetition of the process, there are three possibilities: (1) it may come to an end at some point through the appearance of an equation for the next coefficient, which has only simple roots; (2) it may end on account of the non-existence of continuous partial derivatives of the orders required to carry it on; or (3) the appearance of multiple roots may continue indefinitely. An example of this last is the case where $F(x, y)$ is such a function as $(y - \sin x)^2$. Here

$$F(x, y) = x^2 - 2xy + y^2 - \tfrac{1}{3}x^4 + \tfrac{1}{3}x^3y + \tfrac{2}{45}x^6 - \tfrac{1}{60}x^5y + \cdots,$$

and the coefficient of every term in the assumed solution will be a double root of the equation for determining it. Hence no finite number of terms from the development would enable us to determine the character of the solution, or assert that the origin is not a cusp, an isolated point, or a point common to two distinct branches.

The question of the derivatives of ζ_1 with regard to h, η with regard to t, etc., may be treated by expressing x and y in terms of these new variables, and finding the derivatives of $F(x, y)$ with regard to them, thus reducing the problem to that of Theorem 3.

HARVARD UNIVERSITY,
September 1, 1909.

TABLES OF GALOIS FIELDS OF ORDER LESS THAN 1,000.

BY PROFESSOR W. H. BUSSEY.

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EVERY field of a finite number of marks may be represented as a Galois field of order $s = p^n$, where p^n is a power of a prime. The $GF[p^n]$ is defined uniquely by its order, and is therefore independent of the particular irreducible congruence used in its construction. In each of the following tables the $GF[p^n]$ is constructed by means of a primitive irreducible congruence which appears at the top of the table. The marks of each field are arranged in two tables. In each table each mark appears as a power of a primitive root i, and also as a polynomial in i of degree $k \leqq n-1$. The coefficients in this polynomial are integers reduced modulo p. The mark $Ai^k + Bi^{k-1} + \cdots + Di + E$, $A \neq 0$, is denoted by $AB \cdots DE$, a symbol consisting of its detached coefficients in order. Zero coefficients must not be omitted. This is the usual symbol for a positive integer in the notation of the number system whose base is p. In the first table the marks are arranged according to ascending powers of i. In the second table the marks are arranged so that the symbols $AB \cdots DE$ represent the positive integers in natural order. By means of these two tables it is possible to perform with ease the operations of addition, subtraction, multiplication and division, within the field.

The foregoing paragraph is reprinted from a paper entitled "Galois field tables for $p^n \leqq 169$," published in the BULLETIN, volume 12 (1905), pages 21–38. The present paper is an extension of that work and contains tables made according to the same plan but more compactly printed. The tables for fields of orders 2^8 and 2^9 are printed completely as were all of the tables in the paper mentioned. The other tables are abbreviated according to the following plan: The first of the two tables for a field of order p^n contains the marks i^λ for which $\lambda = 1, 2, 3, \cdots, \nu$ (where ν is the smallest value of λ such that i^λ is equal to one of the numbers $1, 2, 3, \cdots, p-1$) and also the marks i^λ for which λ is a multiple of ν. The table expresses each of these marks as a polynomial in i of degree $k \leqq n-1$. If λ

be any other exponent, the value of i^λ as a polynomial in i may be obtained by writing $\lambda = q\nu + r$, where $q\nu$ is the largest multiple of ν less than λ and consequently r is less than ν. Then $i^\lambda = i^{q\nu+r} = i^{q\nu} \cdot i^r$. The values of $i^{q\nu}$ and i^r may be read off directly from the table and their product obtained mentally because the value of $i^{q\nu}$ is integral. The second of the two tables for a field of order p^n contains the integral marks $1, 2, 3, \cdots, p-1$ and also the marks $\alpha i^\kappa + \beta i^{k-1} + \gamma i^{k-2} + \cdots$ for which $\alpha = 1$. It expresses each of these marks as a power i^λ of the primitive root i. If $\alpha \neq 1$, the mark may be written

$$\alpha\left[i^k + \frac{\beta}{\alpha} i^{k-1} + \frac{\gamma}{\alpha} i^{k-2} + \cdots\right],$$

the divisions by α, modulo p, being made mentally or by means of the ordinary tables of indices for the prime p. The second table gives the value of α and the value of the expression in brackets each as a power of i. Their product is a power of i obtained by adding exponents.

It is convenient to use detached coefficients and denote the mark $\alpha i^k + \beta i^{k-1} + \gamma i^{k-2} + \cdots$ by the symbol $\alpha\beta\gamma\cdots$. These coefficients are separated by commas when any one of them is an integer of more than one digit, that is when the modulus p is an integer of more than one digit. Otherwise it is more convenient not to make such a separation.

This paper and the one published in 1905 contain tables of all Galois fields of order $p^n < 1000$, $n > 1$. This is the limit set by Jacobi in his Canon Arithmeticus, which contains tables of indices for all primes less than 1000, *i. e.*, tables of all Galois fields of prime order $p < 1000$.

Example. The first table for the $GF[17^2]$ contains the marks i^λ for which $\lambda = 1, 2, 3, \cdots, 18$ or $\lambda = 18q$. The table gives the value of each of these marks as a polynomial $\alpha i + \beta$. To express i^{243} as a polynomial of the form $\alpha i + \beta$, write $i^{243} = i^{234}\, i^9$. From the table, $i^{234} = 12$ and $i^9 = 11i + 3$. Consequently $i^{243} = 12(11i + 3) = 132i + 36 = 13i + 2$ because $132 \equiv 13$ and $36 \equiv 2$ modulo 17. The second table contains in order the integral marks $1, 2, 3, \cdots, 16$ and the marks $\alpha i + \beta$ for which $\alpha = 1$. It gives the value of each of these marks as a power of i. The mark $8i + 7$ does not occur in the table because $\alpha = 8$. To express it as a power of i, write $8i + 7 = 8(i + \frac{7}{8}) = 8(i + 3)$ because $\frac{7}{8} \equiv 3$, modulo 17. From the table, $8 = i^{180}$ and $i + 3 = i^{150}$. Therefore $8i + 7 = i^{180+150} = i^{330} = i^{12}$ because $i^{288} = 1$.

$GF[3^5]$, $i^5 \equiv i + 2$, modulo 3. $i^\lambda = \alpha i^4 + \beta i^3 + \gamma i^2 + \delta i + \varepsilon$.

FIRST TABLE.

λ	αβγδε	λ	αβγδε	λ	αβγδε	λ	αβγδε	λ	αβγδε	λ	αβγδε
1	10	21	20222	41	11121	61	1211	81	22112	101	211
2	100	22	2211	42	11222	62	12110	82	21111	102	2110
3	1000	23	22110	43	12202	63	21112	83	11101	103	21100
4	10000	24	21121	44	22002	64	11111	84	11022	104	11021
5	12	25	11201	45	20011	65	11122	85	10202	105	10222
6	120	26	12022	46	101	66	11202	86	2002	106	2202
7	1200	27	20202	47	1010	67	12002	87	20020	107	22020
8	12000	28	2011	48	10100	68	20002	88	221	108	20221
9	20012	29	20110	49	1012	69	11	89	2210	109	2201
10	111	30	1121	50	10120	70	110	90	22100	110	22010
11	1110	31	11210	51	1212	71	1100	91	21021	111	20121
12	11100	32	12112	52	12120	72	11000	92	10201	112	1201
13	11012	33	21102	53	21212	73	10012	93	2022	113	12010
14	10102	34	11011	54	12111	74	102	94	20220	114	20112
15	1002	35	10122	55	21122	75	1020	95	2221	115	1111
16	10020	36	1202	56	11211	76	10200	96	22210	116	11110
17	212	37	12020	57	12122	77	2012	97	22121	117	11112
18	2120	38	20212	58	21202	78	20120	98	21201	118	11102
19	21200	39	2111	59	12011	79	1221	99	12001	119	11002
20	12021	40	21110	60	20122	80	12210	100	20022	120	10002
										121	2

SECOND TABLE.

λ	αβγδε	λ	αβγδε	λ	αβγδε	λ	αβγδε	λ	αβγδε	λ	αβγδε
242	1	198	1021	4	10000	85	10202	64	11111	37	12020
1	10	149	1022	189	10001	199	10210	117	11112	20	12021
69	11	71	1100	120	10002	181	10211	217	11120	26	12022
5	12	227	1101	208	10010	232	10212	41	11121	140	12100
2	100	230	1102	221	10011	150	10220	65	11122	179	12101
46	101	11	1110	73	10012	235	10221	211	11200	219	12102
74	102	115	1111	16	10020	105	10222	25	11201	62	12110
70	110	216	1112	130	10021	72	11000	66	11202	54	12111
10	111	210	1120	166	10022	165	11001	31	11210	32	12112
209	112	30	1121	48	10100	119	11002	56	11211	52	12120
6	120	143	1122	148	10101	228	11010	218	11212	174	12121
138	121	7	1200	14	10102	34	11011	144	11220	57	12122
222	122	112	1201	215	10110	13	11012	202	11221	224	12200
3	1000	36	1202	142	10111	231	11020	42	11222	154	12201
207	1001	139	1210	229	10112	104	11021	8	12000	43	12202
15	1002	61	1211	50	10120	84	11022	99	12001	80	12210
47	1010	51	1212	159	10121	12	11100	67	12002	176	12211
214	1011	223	1220	35	10122	83	11101	113	12010	145	12212
49	1012	79	1221	76	10200	118	11102	59	12011	161	12220
75	1020	160	1222	92	10201	116	11110	212	12012	184	12221
										203	12222

$GF[2^8]$, $i^8 \equiv i^4 + i^3 + i^2 + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \varepsilon i^3 + \zeta i^2 + \eta i + \theta.$

FIRST TABLE.

λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$
1	10	49	10001100	97	10101111	145	1001101
2	100	50	101	98	1000011	146	10011010
3	1000	51	1010	99	10000110	147	101001
4	10000	52	10100	100	10001	148	1010010
5	100000	53	101000	101	100010	149	10100100
6	1000000	54	1010000	102	1000100	150	1010101
7	10000000	55	10100000	103	10001000	151	10101010
8	11101	56	1011101	104	1101	152	1001001
9	111010	57	10111010	105	11010	153	10010010
10	1110100	58	1101001	106	110100	154	111001
11	11101000	59	11010010	107	1101000	155	1110010
12	11001101	60	10111001	108	11010000	156	11100100
13	10000111	61	1101111	109	10111101	157	11010101
14	10011	62	11011110	110	1100111	158	10110111
15	100110	63	10100001	111	11001110	159	1110011
16	1001100	64	1011111	112	10000001	160	11100110
17	10011000	65	10111110	113	11111	161	11010001
18	101101	66	1100001	114	111110	162	10111111
19	1011010	67	11000010	115	1111100	163	1100011
20	10110100	68	10011001	116	11111000	164	11000110
21	1110101	69	101111	117	11101101	165	10010001
22	11101010	70	1011110	118	11000111	166	111111
23	11001001	71	10111100	119	10010011	167	1111110
24	10001111	72	1100101	120	111011	168	11111100
25	11	73	11001010	121	1110110	169	11100101
26	110	74	10001001	122	11101100	170	11010111
27	1100	75	1111	123	11000101	171	10110011
28	11000	76	11110	124	10010111	172	1111011
29	110000	77	111100	125	110011	173	11110110
30	1100000	78	1111000	126	1100110	174	11110001
31	11000000	79	11110000	127	11001100	175	11111111
32	10011101	80	11111101	128	10000101	176	11100011
33	100111	81	11100111	129	10111	177	11011011
34	1001110	82	11010011	130	101110	178	10101011
35	10011100	83	10111011	131	1011100	179	1001011
36	100101	84	1101011	132	10111000	180	10010110
37	1001010	85	11010110	133	1101101	181	110001
38	10010100	86	10110001	134	11011010	182	1100010
39	110101	87	1111111	135	10101001	183	11000100
40	1101010	88	11111110	136	1001111	184	10010101
41	11010100	89	11100001	137	10011110	185	110111
42	10110101	90	11011111	138	100001	186	1101110
43	1110111	91	10100011	139	1000010	187	11011100
44	11101110	92	1011011	140	10000100	188	10100101
45	11000001	93	10110110	141	10101	189	1010111
46	10011111	94	1110001	142	101010	190	10101110
47	100011	95	11100010	143	1010100	191	1000001
48	1000110	96	11011001	144	10101000	192	10000010

$GF[2^8]$, $i^8 \equiv i^4 + i^3 + i^2 + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \varepsilon i^3 + \zeta i^2 + \eta i + \theta.$

FIRST TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$
193	11001	209	10100010	225	100100	241	1011000
194	110010	210	1011001	226	1001000	242	10110000
195	1100100	211	10110010	227	10010000	243	1111101
196	11001000	212	1111001	228	111101	244	11111010
197	10001101	213	11110010	229	1111010	245	11101001
198	111	214	11111001	230	11110100	246	11001111
199	1110	215	11101111	231	11110101	247	10000011
200	11100	216	11000011	232	11110111	248	11011
201	111000	217	10011011	233	11110011	249	110110
202	1110000	218	101011	234	11111011	250	1101100
203	11100000	219	1010110	235	11101011	251	11011000
204	11011101	220	10101100	236	11001011	252	10101101
205	10100111	221	1000101	237	10001011	253	1000111
206	1010011	222	10001010	238	1011	254	10001110
207	10100110	223	1001	239	10110	255	1
208	1010001	224	10010	240	101100		

SECOND TABLE.

λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta$
255	1	138	100001	191	1000001	66	1100001
1	10	101	100010	139	1000010	182	1100010
25	11	47	100011	98	1000011	163	1100011
2	100	225	100100	102	1000100	195	1100100
50	101	36	100101	221	1000101	72	1100101
26	110	15	100110	48	1000110	126	1100110
198	111	33	100111	253	1000111	110	1100111
3	1000	53	101000	226	1001000	107	1101000
223	1001	147	101001	152	1001001	58	1101001
51	1010	142	101010	37	1001010	40	1101010
238	1011	218	101011	179	1001011	84	1101011
27	1100	240	101100	16	1001100	250	1101100
104	1101	18	101101	145	1001101	133	1101101
199	1110	130	101110	34	1001110	186	1101110
75	1111	69	101111	136	1001111	61	1101111
4	10000	29	110000	54	1010000	202	1110000
100	10001	181	110001	208	1010001	94	1110001
224	10010	194	110010	148	1010010	155	1110010
14	10011	125	110011	206	1010011	159	1110011
52	10100	106	110100	143	1010100	10	1110100
141	10101	39	110101	150	1010101	21	1110101
239	10110	249	110110	219	1010110	121	1110110
129	10111	185	110111	189	1010111	43	1110111
28	11000	201	111000	241	1011000	78	1111000
193	11001	154	111001	210	1011001	212	1111001
105	11010	9	111010	19	1011010	229	1111010
248	11011	120	111011	92	1011011	172	1111011
200	11100	77	111100	131	1011100	115	1111100
8	11101	228	111101	56	1011101	243	1111101
76	11110	114	111110	70	1011110	167	1111110
113	11111	166	111111	64	1011111	87	1111111
5	100000	6	1000000	30	1100000	7	10000000

GF $[2^8]$, $i^8 \equiv i^4 + i^3 + i^2 + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \varepsilon i^3 + \zeta i^2 + \eta i + \theta.$

SECOND TABLE.—*Continued.*

λ	αβγδεζηθ	λ	αβγδεζηθ	λ	αβγδεζηθ	λ	αβγδεζηθ
112	10000001	63	10100001	45	11000001	89	11100001
192	10000010	209	10100010	67	11000010	95	11100010
247	10000011	91	10100011	216	11000011	176	11100011
140	10000100	149	10100100	183	11000100	156	11100100
128	10000101	188	10100101	123	11000101	169	11100101
99	10000110	207	10100110	164	11000110	160	11100110
13	10000111	205	10100111	118	11000111	81	11100111
103	10001000	144	10101000	196	11001000	11	11101000
74	10001001	135	10101001	23	11001001	245	11101001
222	10001010	151	10101010	73	11001010	22	11101010
237	10001011	178	10101011	236	11001011	235	11101011
49	10001100	220	10101100	127	11001100	122	11101100
197	10001101	252	10101101	12	11001101	117	11101101
254	10001110	190	10101110	111	11001110	44	11101110
24	10001111	97	10101111	246	11001111	215	11101111
227	10010000	242	10110000	108	11010000	79	11110000
165	10010001	86	10110001	161	11010001	174	11110001
153	10010010	211	10110010	59	11010010	213	11110010
119	10010011	171	10110011	82	11010011	233	11110011
38	10010100	20	10110100	41	11010100	230	11110100
184	10010101	42	10110101	157	11010101	231	11110101
180	10010110	93	10110110	85	11010110	173	11110110
124	10010111	158	10110111	170	11010111	232	11110111
17	10011000	132	10111000	251	11011000	116	11111000
68	10011001	60	10111001	96	11011001	214	11111001
146	10011010	57	10111010	134	11011010	244	11111010
217	10011011	83	10111011	177	11011011	234	11111011
35	10011100	71	10111100	187	11011100	168	11111100
32	10011101	109	10111101	204	11011101	80	11111101
137	10011110	65	10111110	62	11011110	88	11111110
46	10011111	162	10111111	90	11011111	175	11111111
55	10100000	31	11000000	203	11100000		

GF $[17^2]$, $i^2 \equiv i + 14$, modulo 17. $i^\lambda = \alpha i + \beta.$

FIRST TABLE.

λ	α , β	λ	α , β	λ	α , β	λ	α , β
1	1 , 0	12	7 , 6	108	15	288	1
2	1 , 14	13	13 , 13	126	11	252	2
3	15 , 14	14	9 , 12	144	16	18	3
4	12 , 6	15	4 , 7	162	14	216	4
5	1 , 15	16	11 , 5	180	8	90	5
6	16 , 14	17	16 , 1	198	7	270	6
7	13 , 3	18	3	216	4	198	7
8	16 , 12	36	9	234	12	180	8
9	11 , 3	54	10	252	2	36	9
10	14 , 1	72	13	270	6	54	10
11	15 , 9	90	5	288	1	126	11

SECOND TABLE.

λ	α , β	λ	α , β	λ	α , β
234	12	87	1 , 6		
72	13	266	1 , 7		
162	14	171	1 , 8		
108	15	58	1 , 9		
144	16	183	1 , 10		
1	1 , 0	136	1 , 11		
229	1 , 1	223	1 , 12		
178	1 , 2	102	1 , 13		
150	1 , 3	2	1 , 14		
191	1 , 4	5	1 , 15		
152	1 , 5	161	1 , 16		

$GF\ [7^3]$, $i^3 \equiv i + 5$, modulo 7. $i\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	αβγ	λ	αβγ	λ	αβγ
1	10	22	561	43	11
2	100	23	664	44	110
3	15	24	632	45	115
4	150	25	312	46	165
5	515	26	151	47	665
6	134	27	525	48	642
7	355	28	234	49	412
8	511	29	363	50	166
9	164	30	661	51	605
10	655	31	602	52	42
11	542	32	12	53	420
12	404	33	120	54	246
13	16	34	215	55	413
14	160	35	103	56	106
15	615	36	45	57	5
16	142	37	450	114	4
17	435	38	546	171	6
18	326	39	444	228	2
19	221	40	416	285	3
20	233	41	136	342	1
21	353	42	305		

SECOND TABLE.

λ	αβγ	λ	αβγ	λ	αβγ
342	1	267	111	78	141
228	2	218	112	16	142
285	3	194	113	307	143
114	4	133	114	64	144
57	5	45	115	195	145
171	6	201	116	148	146
1	10	33	120	4	150
43	11	86	121	26	151
32	12	181	122	142	152
264	13	168	123	82	153
280	14	277	124	323	154
3	15	268	125	134	155
13	16	283	126	296	156
2	100	265	130	14	160
240	101	290	131	312	161
222	102	75	132	186	162
35	103	293	133	245	163
99	104	6	134	9	164
202	105	219	135	46	165
56	106	41	136	50	166
44	110	281	140		

$GF\ [19^2]$, $i^2 \equiv i + 17$, modulo 19. $i\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α , β	λ	α , β	λ	α , β
1	1 , 0	14	4 , 2	160	9
2	1 , 17	15	6 , 11	180	18
3	18 , 17	16	17 , 7	200	17
4	16 , 2	17	5 , 4	220	15
5	18 , 6	18	9 , 9	240	11
6	5 , 2	19	18 , 1	260	3
7	7 , 9	20	2	280	6
8	16 , 5	40	4	300	12
9	2 , 6	60	8	320	5
10	8 ; 15	80	16	340	10
11	4 , 3	100	13	360	1
12	7 , 11	120	7		
13	18 , 5	140	14		

SECOND TABLE.

λ	α , β	λ	α , β	λ	α , β
360	1	140	14	46	1 , 8
20	2	220	15	310	1 , 9
260	3	80	16	334	1 , 10
40	4	200	17	288	1 , 11
320	5	180	18	284	1 , 12
280	6	1	1 , 0	185	1 , 13
120	7	218	1 , 1	193	1 , 14
60	8	183	1 , 2	331	1 , 15
160	9	349	1 , 3	57	1 , 16
340	10	247	1 , 4	2	1 , 17
240	11	95	1 , 5	199	1 , 18
300	12	176	1 , 6		
100	13	252	1 , 7		

$GF[2^9]$, $i^9 \equiv i^5 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \varepsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa.$

FIRST TABLE.

λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ
1	10	49	111111111	97	1100110	145	101011011
2	100	50	11100001	98	11001100	146	110101001
3	1000	51	111000010	99	110011000	147	1001101
4	10000	52	10011011	100	101111	148	10011010
5	100000	53	100110110	101	1011110	149	100110100
6	1000000	54	101110011	102	10111100	150	101110111
7	10000000	55	111111001	103	101111000	151	111110001
8	100000000	56	11101101	104	111101111	152	11111101
9	100011111	57	111011010	105	11000001	153	111111010
10	100100001	58	10101011	106	110000010	154	11101011
11	101011101	59	101010110	107	11011	155	111010110
12	110100101	60	110110011	108	110110	156	10110011
13	1010101	61	1111001	109	1101100	157	101100110
14	10101010	62	11110010	110	11011000	158	111010011
15	101010100	63	111100100	111	110110000	159	10111001
16	110110111	64	11010111	112	1111111	160	101110010
17	1110001	65	110101110	113	11111110	161	111111011
18	11100010	66	1000011	114	111111100	162	11101001
19	111000100	67	10000110	115	11100111	163	111010010
20	10010111	68	100001100	116	111001110	164	10111011
21	100101110	69	100000111	117	10000011	165	101110110
22	101000011	70	100010001	118	100000110	166	111110011
23	110011001	71	100111101	119	100010011	167	11111001
24	101101	72	101100101	120	100111001	168	111110010
25	1011010	73	111010101	121	101101101	169	11111011
26	10110100	74	10110101	122	111000101	170	111110110
27	101101000	75	101101010	123	10010101	171	11110011
28	111001111	76	111001011	124	100101010	172	111100110
29	10000001	77	10001001	125	101001011	173	11010011
30	100000010	78	100010010	126	110001001	174	110100110
31	100011011	79	100111011	127	1101	175	1010011
32	100101001	80	101101001	128	11010	176	10100110
33	101001101	81	111001101	129	110100	177	101001100
34	110000101	82	10000101	130	1101000	178	110000111
35	10101	83	100001010	131	11010000	179	10001
36	101010	84	100001011	132	110100000	180	100010
37	1010100	85	100001001	133	1011111	181	1000100
38	10101000	86	100001101	134	10111110	182	10001000
39	101010000	87	100000101	135	101111100	183	100010000
40	110111111	88	100010101	136	111100111	184	100111111
41	1100001	89	100110101	137	11010001	185	101100001
42	11000010	90	101110101	138	110100010	186	111011101
43	110000100	91	111110101	139	1011011	187	10100101
44	10111	92	11110101	140	10110110	188	101001010
45	101110	93	111101010	141	101101100	189	110001011
46	1011100	94	11001011	142	111000111	190	1001
47	10111000	95	110010110	143	10010001	191	10010
48	101110000	96	110011	144	100100010	192	100100

$GF[2^8]$, $i^8 \equiv i^6 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \varepsilon i^3 + \zeta i^2 + \eta i + \theta$.

FIRST TABLE.—*Continued.*

λ	αβγδεζηθ	λ	αβγδεζηθ	λ	αβγδεζηθ	λ	αβγδεζηθ
193	10010000	241	10110001	289	100100100	337	100111010
194	10010000	242	101100010	290	101010111	338	101101011
195	100100000	243	111011011	291	110110001	339	111001001
196	101011111	244	10101001	292	1111101	340	10001101
197	110100001	245	101010010	293	11111010	341	100011010
198	1011101	246	110111011	294	111110100	342	100101011
199	10111010	247	1101001	295	11110111	343	101001001
200	101110100	248	11010010	296	111101110	344	110001101
201	111110111	249	110100100	297	11000011	345	101
202	11110001	250	1010111	298	110000110	346	1010
203	111100010	251	10101110	299	10011	347	10100
204	11011011	252	101011100	300	100110	348	101000
205	110110110	253	110100111	301	1001100	349	1010000
206	1110011	254	1010001	302	10011000	350	10100000
207	11100110	255	10100010	303	100110000	351	101000000
208	111001100	256	101000100	304	101111111	352	110011111
209	10000111	257	110010111	305	111100001	353	100001
210	100001110	258	110001	306	11011101	354	1000010
211	100000011	259	1100010	307	110111010	355	10000100
212	100011001	260	11000100	308	1101011	356	100001000
213	100101101	261	110001000	309	11010110	357	100001111
214	101000101	262	1111	310	110101100	358	100000001
215	110010101	263	11110	311	1000111	359	100011101
216	110101	264	111100	312	10001110	360	100100101
217	1101010	265	1111000	313	100011100	361	101010101
218	11010100	266	11110000	314	100100111	362	110110101
219	110101000	267	111100000	315	101010001	363	1110101
220	1001111	268	11011111	316	110111101	364	11101010
221	10011110	269	110111110	317	1100101	365	111010100
222	100111100	270	1100011	318	11001010	366	10110111
223	101100111	271	11000110	319	110010100	367	101101110
224	111010001	272	110001100	320	110111	368	111000011
225	10111101	273	111	321	1101110	369	10011001
226	101111010	274	1110	322	11011100	370	100110010
227	111101011	275	11100	323	110111000	371	101111011
228	11001001	276	111000	324	1101111	372	111101001
229	110010010	277	1110000	325	11011110	373	11001101
230	111011	278	11100000	326	110111100	374	110011010
231	1110110	279	111000000	327	1100111	375	101011
232	11101100	280	10011111	328	11001110	376	1010110
233	111011000	281	100111110	329	110011100	377	10101100
234	10101111	282	101100011	330	100111	378	101011000
235	101011110	283	111011001	331	1001110	379	110101111
236	110100011	284	10101101	332	10011100	380	1000001
237	1011001	285	101011010	333	100111000	381	10000010
238	10110010	286	110101011	334	101101111	382	100000100
239	101100100	287	1001001	335	111000001	383	100010111
240	111010111	288	10010010	336	10011101	384	100110001

$GF[2^9]$, $i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \varepsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa.$

FIRST TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta\kappa$	λ	$\alpha\beta\gamma\delta\varepsilon\zeta\eta\theta\kappa$
385	101111101	417	111010000	449	1101101	481	1000101
386	111100101	418	10111111	450	11011010	482	10001010
387	11010101	419	101111110	451	110110100	483	100010100
388	110101010	420	111100011	452	1110111	484	100110111
389	1001011	421	11011001	453	11101110	485	101110001
390	10010110	422	110110010	454	111011100	486	111111101
391	100101100	423	1111011	455	10100111	487	11100101
392	101000111	424	11110110	456	101001110	488	111001010
393	110010001	425	111101100	457	110000011	489	10001011
394	111101	426	11000111	458	11001	490	100010110
395	1111010	427	110001110	459	110010	491	100110011
396	11110100	428	11	460	1100100	492	101111001
397	111101000	429	110	461	11001000	493	111101101
398	11001111	430	1100	462	110010000	494	11000101
399	110011110	431	11000	463	111111	495	110001010
400	100011	432	110000	464	1111110	496	1011
401	1000110	433	1100000	465	11111100	497	10110
402	10001100	434	11000000	466	111111000	498	101100
403	100011000	435	110000000	467	11101111	499	1011000
404	100101111	436	11111	468	111011110	500	10110000
405	101000001	437	111110	469	10100011	501	101100000
406	110011101	438	1111100	470	101000110	502	111011111
407	100101	439	11111000	471	110010011	503	10100001
408	1001010	440	111110000	472	111001	504	101000010
409	10010100	441	11111111	473	1110010	505	110011011
410	100101000	442	111111110	474	11100100	506	101001
411	101001111	443	11100011	475	111001000	507	1010010
412	110000001	444	111000110	476	10001111	508	10100100
413	11101	445	10010011	477	100011110	509	101001000
414	111010	446	100100110	478	100100011	510	110001111
415	1110100	447	101010011	479	101011001	511	1
416	11101000	448	110111001	480	110101101		

$GF[2^9]$, $i^9 \equiv i^8 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^8 + \beta i^7 + \gamma i^6 + \delta i^5 + \varepsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa$.

SECOND TABLE.

λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ
511	1	353	100001	380	1000001	41	1100001
1	10	180	100010	354	1000010	259	1100010
428	11	400	100011	66	1000011	270	1100011
2	100	192	100100	181	1000100	460	1100100
345	101	407	100101	481	1000101	317	1100101
429	110	300	100110	401	1000110	97	1100110
273	111	330	100111	311	1000111	327	1100111
3	1000	348	101000	193	1001000	130	1101000
190	1001	506	101001	287	1001001	247	1101001
346	1010	36	101010	408	1001010	217	1101010
496	1011	375	101011	389	1001011	308	1101011
430	1100	498	101100	301	1001100	109	1101100
127	1101	24	101101	147	1001101	449	1101101
274	1110	45	101110	331	1001110	321	1101110
262	1111	100	101111	220	1001111	324	1101111
4	10000	432	110000	349	1010000	277	1110000
179	10001	258	110001	254	1010001	17	1110001
191	10010	459	110010	507	1010010	473	1110010
299	10011	96	110011	175	1010011	206	1110011
347	10100	129	110100	37	1010100	415	1110100
35	10101	216	110101	13	1010101	363	1110101
497	10110	108	110110	376	1010110	231	1110110
44	10111	320	110111	250	1010111	452	1110111
431	11000	276	111000	499	1011000	265	1111000
458	11001	472	111001	237	1011001	61	1111001
128	11010	414	111010	25	1011010	395	1111010
107	11011	230	111011	139	1011011	423	1111011
275	11100	264	111100	46	1011100	438	1111100
413	11101	394	111101	198	1011101	292	1111101
263	11110	437	111110	101	1011110	464	1111110
436	11111	463	111111	133	1011111	112	1111111
5	100000	6	1000000	433	1100000	7	10000000

$GF[2^8]$, $i^8 \equiv i^5 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^7 + \gamma i^6 + \delta i^5 + \epsilon i^4 + \zeta i^3 + \eta i^2 + \theta i + \kappa.$

SECOND TABLE.—*Continued.*

λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ	λ	αβγδεζηθκ
29	10000001	241	10110001	50	11100001	70	100010001
381	10000010	238	10110010	18	11100010	78	100010010
117	10000011	156	10110011	443	11100011	119	100010011
355	10000100	26	10110100	474	11100100	483	100010100
82	10000101	74	10110101	487	11100101	88	100010101
67	10000110	140	10110110	207	11100110	490	100010110
209	10000111	366	10110111	115	11100111	383	100010111
182	10001000	47	10111000	416	11101000	403	100011000
77	10001001	159	10111001	162	11101001	212	100011001
482	10001010	199	10111010	364	11101010	341	100011010
489	10001011	164	10111011	154	11101011	31	100011011
402	10001100	102	10111100	232	11101100	313	100011100
340	10001101	225	10111101	56	11101101	359	100011101
312	10001110	134	10111110	453	11101110	477	100011110
476	10001111	418	10111111	467	11101111	9	100011111
194	10010000	434	11000000	266	11110000	195	100100000
143	10010001	105	11000001	202	11110001	10	100100001
288	10010010	42	11000010	62	11110010	144	100100010
445	10010011	297	11000011	171	11110011	478	100100011
409	10010100	260	11000100	396	11110100	289	100100100
123	10010101	494	11000101	92	11110101	360	100100101
390	10010110	271	11000110	424	11110110	446	100100110
20	10010111	426	11000111	295	11110111	314	100100111
302	10011000	461	11001000	439	11111000	410	100101000
369	10011001	228	11001001	167	11111001	32	100101001
148	10011010	318	11001010	293	11111010	124	100101010
52	10011011	94	11001011	169	11111011	342	100101011
332	10011100	98	11001100	465	11111100	391	100101100
336	10011101	373	11001101	152	11111101	213	100101101
221	10011110	328	11001110	113	11111110	21	100101110
280	10011111	398	11001111	441	11111111	404	100101111
350	10100000	131	11010000	8	100000000	303	100110000
503	10100001	137	11010001	358	100000001	384	100110001
255	10100010	248	11010010	30	100000010	370	100110010
469	10100011	173	11010011	211	100000011	491	100110011
508	10100100	218	11010100	382	100000100	149	100110100
187	10100101	387	11010101	87	100000101	89	100110101
176	10100110	309	11010110	118	100000110	53	100110110
455	10100111	64	11010111	69	100000111	484	100110111
38	10101000	110	11011000	356	100001000	333	100111000
244	10101001	421	11011001	85	100001001	120	100111001
14	10101010	450	11011010	83	100001010	337	100111010
58	10101011	204	11011011	84	100001011	79	100111011
377	10101100	322	11011100	68	100001100	222	100111100
284	10101101	306	11011101	86	100001101	71	100111101
251	10101110	325	11011110	210	100001110	281	100111110
234	10101111	268	11011111	357	100001111	184	100111111
500	10110000	278	11100000	183	100010000	351	101000000

$GF[2^8]$, $i^8 \equiv i^6 + i^4 + i^3 + i^2 + i + 1$, modulo 2.
$i^\lambda = \alpha i^7 + \beta i^6 + \gamma i^5 + \delta i^4 + \varepsilon i^3 + \zeta i^2 + \eta i + \theta$.

SECOND TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta\eta\theta$
405	101000001	485	101110001	197	110100001	224	111010001
504	101000010	160	101110010	138	110100010	163	111010010
22	101000011	54	101110011	236	110100011	158	111010011
256	101000100	200	101110100	249	110100100	365	111010100
214	101000101	90	101110101	12	110100101	73	111010101
470	101000110	165	101110110	174	110100110	155	111010110
392	101000111	150	101110111	253	110100111	240	111010111
509	101001000	103	101111000	219	110101000	233	111011000
343	101001001	492	101111001	146	110101001	283	111011001
188	101001010	226	101111010	388	110101010	57	111011010
125	101001011	371	101111011	286	110101011	243	111011011
177	101001100	135	101111100	310	110101100	454	111011100
33	101001101	385	101111101	480	110101101	186	111011101
456	101001110	419	101111110	65	110101110	468	111011110
411	101001111	304	101111111	379	110101111	502	111011111
39	101010000	435	110000000	111	110110000	267	111100000
315	101010001	412	110000001	291	110110001	305	111100001
245	101010010	106	110000010	422	110110010	203	111100010
447	101010011	457	110000011	60	110110011	420	111100011
15	101010100	43	110000100	451	110110100	63	111100100
361	101010101	34	110000101	362	110110101	386	111100101
59	101010110	298	110000110	205	110110110	172	111100110
290	101010111	178	110000111	16	110110111	136	111100111
378	101011000	261	110001000	323	110111000	397	111101000
479	101011001	126	110001001	448	110111001	372	111101001
285	101011010	495	110001010	307	110111010	93	111101010
145	101011011	189	110001011	246	110111011	227	111101011
252	101011100	272	110001100	326	110111100	425	111101100
11	101011101	344	110001101	316	110111101	493	111101101
235	101011110	427	110001110	269	110111110	296	111101110
196	101011111	510	110001111	40	110111111	104	111101111
501	101100000	462	110010000	279	111000000	440	111110000
185	101100001	393	110010001	335	111000001	151	111110001
242	101100010	229	110010010	51	111000010	168	111110010
282	101100011	471	110010011	368	111000011	166	111110011
239	101100100	319	110010100	19	111000100	294	111110100
72	101100101	215	110010101	122	111000101	91	111110101
157	101100110	95	110010110	444	111000110	170	111110110
223	101100111	257	110010111	142	111000111	201	111110111
27	101101000	99	110011000	475	111001000	466	111111000
80	101101001	23	110011001	339	111001001	55	111111001
75	101101010	374	110011010	488	111001010	153	111111010
338	101101011	505	110011011	76	111001011	161	111111011
141	101101100	329	110011100	208	111001100	114	111111100
121	101101101	406	110011101	81	111001101	486	111111101
367	101101110	399	110011110	116	111001110	442	111111110
334	101101111	352	110011111	28	111001111	49	111111111
48	101110000	132	110100000	417	111010000		

$GF[23^2]$, $i^2 \equiv i + 16$, modulo 23. $i^\lambda = \alpha i + \beta$.

First Table.

λ	α, β	λ	α, β	λ	α, β
1	1, 0	16	19,19	192	12
2	1,16	17	15, 5	216	15
3	17,16	18	20,10	240	13
4	10,19	19	7,21	264	22
5	6,22	20	5,20	288	16
6	5, 4	21	2,11	312	20
7	9,11	22	13, 9	336	2
8	20, 6	23	22, 1	360	14
9	3,21	24	7	384	6
10	1, 2	48	3	408	19
11	3,16	72	21	432	18
12	19, 2	96	9	456	11
13	21, 5	120	17	480	8
14	3,14	144	4	504	10
15	17, 2	168	5	528	1

Second Table.

λ	α, β	λ	α, β	λ	α, β
528	1	288	16	329	1, 8
336	2	120	17	469	1, 9
48	3	432	18	366	1,10
144	4	408	19	132	1,11
168	5	312	20	234	1,12
384	6	72	21	491	1,13
24	7	264	22	439	1,14
480	8	1	1, 0	423	1,15
96	9	136	1, 1	2	1,16
504	10	10	1, 2	213	1,17
456	11	523	1, 3	28	1,18
192	12	380	1, 4	149	1,19
240	13	411	1, 5	494	1,20
360	14	310	1, 6	224	1,21
216	15	489	1, 7	287	1,22

$GF[5^4]$, $i^4 \equiv i^3 + i + 2$, modulo 5. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

First Table.

λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$	λ	$\alpha\beta\gamma\delta$
1	10	28	3311	55	323	82	3434	109	1020	136	3044
2	100	29	1141	56	3230	83	2321	110	1212	137	3421
3	1000	30	2422	57	331	84	234	111	3132	138	2241
4	1012	31	1244	58	3310	85	2340	112	4301	139	4434
5	1132	32	3402	59	1131	86	424	113	2003	140	3333
6	2332	33	2001	60	2322	87	4240	114	2004	141	1311
7	344	34	2034	61	244	88	1443	115	2014	142	4122
8	3440	35	2314	62	2440	89	442	116	2114	143	213
9	2431	36	114	63	1424	90	4420	117	3114	144	2130
10	1334	37	1140	64	202	91	3243	118	4121	145	3324
11	4302	38	2412	65	2020	92	411	119	203	146	1221
12	2013	39	1144	66	2224	93	4110	120	2030	147	3222
13	2104	40	2402	67	4214	94	143	121	2324	148	201
14	3014	41	1044	68	1133	95	1430	122	214	149	2010
15	3121	42	1402	69	2342	96	312	123	2140	150	2124
16	4241	43	32	70	444	97	3120	124	3424	151	3214
17	1403	44	320	71	4440	98	4231	125	2221	152	121
18	42	45	3200	72	3443	99	1303	126	4234	153	1210
19	420	46	31	73	2411	100	4042	127	1333	154	3112
20	4200	47	310	74	1134	101	4413	128	4342	155	4101
21	1043	48	3100	75	2302	102	3123	129	2413	156	3
22	1442	49	4031	76	44	103	4211	130	1104	312	4
23	432	50	4303	77	440	104	1103	131	2002	468	2
24	4320	51	2023	78	4400	105	2042	132	2044	624	1
25	2243	52	2204	79	3043	106	2444	133	2414		
26	4404	53	4014	80	3411	107	1414	134	1114		
27	3033	54	4133	81	2141	108	102	135	2102		

$GF[5^4]$, $i^4 \equiv i^3 + i + 2$, modulo 5. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

SECOND TABLE.

λ	αβγδ	λ	αβγδ	λ	αβγδ	λ	αβγδ	λ	αβγδ	λ	αβγδ
624	1	475	133	276	1040	413	1142	31	1244	231	1401
468	2	299	134	365	1041	613	1143	332	1300	42	1402
156	3	512	140	190	1042	39	1144	291	1301	17	1403
312	4	523	141	21	1043	516	1200	169	1302	467	1404
1	10	240	142	41	1044	196	1201	99	1303	524	1410
388	11	94	143	390	1100	362	1202	500	1304	216	1411
514	12	404	144	338	1101	323	1203	399	1310	277	1412
330	13	3	1000	208	1102	424	1204	141	1311	239	1413
511	14	287	1001	104	1103	153	1210	306	1312	107	1414
2	100	270	1002	130	1104	186	1211	550	1313	241	1420
220	101	189	1003	383	1110	110	1212	328	1314	225	1421
108	102	269	1004	608	1111	440	1213	279	1320	366	1422
304	103	221	1010	222	1112	579	1214	438	1321	619	1423
275	104	495	1011	281	1113	218	1220	548	1322	63	1424
389	110	4	1012	134	1114	146	1221	237	1323	95	1430
382	111	412	1013	526	1120	262	1222	410	1324	559	1431
525	112	207	1014	451	1121	585	1223	476	1330	191	1432
401	113	109	1020	496	1122	622	1224	540	1331	454	1433
36	114	261	1021	294	1123	336	1230	272	1332	430	1434
515	120	288	1022	181	1124	194	1231	127	1333	405	1440
152	121	482	1023	402	1130	289	1232	10	1334	162	1441
217	122	361	1024	59	1131	229	1233	300	1340	22	1442
335	123	305	1030	5	1132	285	1234	379	1341	88	1443
564	124	547	1031	68	1133	565	1240	605	1342	615	1444
331	130	271	1032	74	1134	570	1241	592	1343		
398	131	604	1033	37	1140	483	1242	415	1344		
278	132	168	1034	29	1141	165	1243	513	1400		

$GF[3^6]$, $i^6 \equiv i + 1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \varepsilon i + \zeta$.

FIRST TABLE.

λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ
1	10	14	12100	27	11121	40	120220	53	101102
2	100	15	121000	28	111210	41	202211	54	11001
3	1000	16	210011	29	112111	42	22102	55	110010
4	10000	17	100102	30	121121	43	221020	56	100111
5	100000	18	1001	31	211221	44	210222	57	1121
6	11	19	10010	32	112202	45	102212	58	11210
7	110	20	100100	33	122001	46	22101	59	112100
8	1100	21	1011	34	220021	47	221010	60	121011
9	11000	22	10110	35	200202	48	210122	61	210121
10	110000	23	101100	36	2012	49	101212	62	101202
11	100011	24	11011	37	20120	50	12101	63	12001
12	121	25	110110	38	201200	51	121010	64	120010
13	1210	26	101111	39	12022	52	210111	65	200111

GF [3⁶], $i^6 \equiv i+1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \varepsilon i + \zeta$.

FIRST TABLE.—*Continued.*

λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ
66	1102	114	220022	162	11122	210	2221	258	12102
67	11020	115	200212	163	111220	211	22210	259	121020
68	110200	116	2112	164	112211	212	222100	260	210211
69	102011	117	21120	165	122121	213	221022	261	102102
70	20121	118	211200	166	221221	214	210212	262	21001
71	201210	119	112022	167	212202	215	102112	263	210010
72	12122	120	120201	168	122012	216	21101	264	100122
73	121220	121	202021	169	220101	217	211010	265	1201
74	212211	122	20202	170	201002	218	110122	266	12010
75	122102	123	202020	171	10012	219	101201	267	120100
76	221001	124	20222	172	100120	220	12021	268	201011
77	210002	125	202220	173	1211	221	120210	269	10102
78	100012	126	22222	174	12110	222	202111	270	101020
79	101	127	222220	175	121100	223	21102	271	10211
80	1010	128	222222	176	211011	224	211020	272	102110
81	10100	129	222212	177	110102	225	110222	273	21111
82	101000	130	222112	178	101001	226	102201	274	211110
83	10011	131	221112	179	10021	227	22021	275	111122
84	100110	132	211112	180	100210	228	220210	276	111201
85	1111	133	111112	181	2111	229	202122	277	112021
86	11110	134	111101	182	21110	230	21212	278	120221
87	111100	135	111021	183	211100	231	212120	279	202221
88	111011	136	110221	184	111022	232	121222	280	22202
89	110121	137	102221	185	110201	233	212201	281	222020
90	101221	138	22221	186	102021	234	122002	282	220222
91	12221	139	222210	187	20221	235	220001	283	202212
92	122210	140	222122	188	202210	236	200002	284	22112
93	222111	141	221212	189	22122	237	12	285	221120
94	221102	142	212112	190	221220	238	120	286	211222
95	211012	143	121112	191	212222	239	1200	287	112212
96	110112	144	211101	192	122212	240	12000	288	122101
97	101101	145	111002	193	222101	241	120000	289	221021
98	11021	146	110001	194	221002	242	200011	290	210202
99	110210	147	100021	195	210012	243	102	291	102012
100	102111	148	221	196	100112	244	1020	292	20101
101	21121	149	2210	197	1101	245	10200	293	201010
102	211210	150	22100	198	11010	246	102000	294	10122
103	112122	151	221000	199	110100	247	20011	295	101220
104	121201	152	210022	200	101011	248	200110	296	12211
105	212021	153	100212	201	10121	249	1122	297	122110
106	120202	154	2101	202	101210	250	11220	298	221111
107	202001	155	21010	203	12111	251	112200	299	211102
108	20002	156	210100	204	121110	252	122011	300	111012
109	200020	157	101022	205	211111	253	220121	301	110101
110	222	158	10201	206	111102	254	201202	302	101021
111	2220	159	102010	207	111001	255	12012	303	10221
112	22200	160	20111	208	110021	256	120120	304	102210
113	222000	161	201110	209	100221	257	201211	305	22111

GF [3⁶], $i^6 \equiv i + 1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \varepsilon i + \zeta$.

First Table.—*Continued.*

λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ
306	221110	318	121200	330	2021	342	211211	354	220110
307	211122	319	212011	331	20210	343	112102	355	201122
308	111212	320	120102	332	202100	344	121001	356	11212
309	112101	321	201001	333	21022	345	210021	357	112120
310	121021	322	10002	334	210220	346	100202	358	121211
311	210221	323	100020	335	102222	347	2001	359	212121
312	102202	324	211	336	22201	348	20010	360	121202
313	22001	325	2110	337	222010	349	200100	361	212001
314	220010	326	21100	338	220122	350	1022	362	120002
315	200122	327	211000	339	201212	351	10220	363	200001
316	1212	328	110022	340	12112	352	102200	364	2
317	12120	329	100201	341	121120	353	22011		

Second Table.

λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ	λ	αβγδεζ
728	1	316	1212	591	11012	340	12112	153	100212
1	10	689	1220	67	11020	317	12120	612	100220
6	11	480	1221	98	11021	594	12121	209	100221
237	12	545	1222	717	11022	72	12122	429	100222
2	100	4	10000	476	11100	690	12200	82	101000
79	101	472	10001	644	11101	587	12201	178	101001
243	102	322	10002	700	11102	580	12202	471	101002
7	110	19	10010	86	11110	481	12210	487	101010
474	111	83	10011	490	11111	296	12211	200	101011
512	112	171	10012	502	11112	465	12212	485	101012
238	120	712	10020	575	11120	546	12220	270	101020
12	121	179	10021	27	11121	91	12221	302	101021
688	122	611	10022	162	11122	637	12222	157	101022
3	1000	81	10100	514	11200	5	100000	23	101100
18	1001	486	10101	406	11201	600	100001	97	101101
711	1002	269	10102	410	11202	727	100002	53	101102
80	1010	22	10110	58	11210	473	100010	489	101110
21	1011	488	10111	553	11211	11	100011	26	101111
694	1012	551	10112	356	11212	78	100012	643	101112
244	1020	695	10120	250	11220	323	100020	552	101120
400	1021	201	10121	648	11221	147	100021	647	101121
350	1022	294	10122	669	11222	606	100022	405	101122
8	1100	245	10200	240	12000	20	100100	696	101200
197	1101	158	10201	63	12001	399	100101	219	101201
66	1102	656	10202	626	12002	17	100102	62	101202
475	1110	401	10210	266	12010	84	100110	202	101210
85	1111	271	10211	697	12011	56	100111	593	101211
574	1112	434	10212	255	12012	196	100112	49	101212
513	1120	351	10220	519	12020	172	100120	295	101220
57	1121	303	10221	220	12021	479	100121	90	101221
249	1122	524	10222	39	12022	264	100122	586	101222
239	1200	9	11000	14	12100	713	100200	246	102000
265	1201	54	11001	50	12101	329	100201	534	102001
518	1202	677	11002	258	12102	346	100202	685	102002
13	1210	198	11010	174	12110	180	100210	159	102010
173	1211	24	11011	203	12111	679	100211	69	102011

$GF\ [3^6]$, $i^6 \equiv i + 1$, modulo 3. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \zeta$.

SECOND TABLE.—*Continued.*

λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$	λ	$\alpha\beta\gamma\delta\epsilon\zeta$
291	102012	592	110120	494	111221	380	120022	318	121200
657	102020	89	110121	457	111222	267	120100	104	121201
186	102021	218	110122	515	112000	654	120101	360	121202
632	102022	68	110200	558	112001	320	120102	595	121210
402	102100	185	110201	440	112002	698	120110	358	121211
618	102101	533	110202	407	112010	408	120111	723	121212
261	102102	99	110210	577	112011	675	120112	73	121220
272	102110	702	110211	653	112012	256	120120	506	121221
100	102111	617	110212	411	112020	578	120121	232	121222
215	102112	718	110220	277	112021	624	120122	691	122000
435	102120	136	110221	119	112022	520	120200	33	122001
703	102121	225	110222	59	112100	120	120201	234	122002
621	102122	477	111000	309	112101	106	120202	588	122010
352	102200	207	111001	343	112102	221	120210	252	122011
226	102201	145	111002	554	112110	412	120211	168	122012
312	102202	645	111010	29	112111	425	120212	581	122020
304	102210	88	111011	530	112112	40	120220	459	122021
719	102211	300	111012	357	112120	278	120221	540	122022
45	102212	701	111020	505	112121	416	120222	482	122100
525	102220	135	111021	103	112122	15	121000	288	122101
137	102221	184	111022	251	112200	344	121001	75	122102
335	102222	87	111100	458	112201	725	121002	297	122110
10	110000	134	111101	32	112202	51	121010	650	122111
146	110001	206	111102	649	112210	60	121011	395	122112
599	110002	491	111110	164	112211	469	121012	466	122120
55	110010	492	111111	287	112212	259	121020	165	122121
478	110011	133	111112	670	112220	310	121021	706	122122
398	110012	503	111120	495	112221	683	121022	547	122200
678	110020	493	111121	662	112222	175	121100	663	122201
208	110021	275	111122	241	120000	531	121101	508	122202
328	110022	576	111200	441	120001	597	121102	92	122210
199	110100	276	111201	362	120002	204	121110	671	122211
301	110101	557	111202	64	120010	555	121111	192	122212
177	110102	28	111210	516	120011	143	121112	638	122220
25	110110	504	111211	709	120012	341	121120	496	122221
646	110111	308	111212	627	120020	30	121121	569	122222
96	110112	163	111220	559	120021	438	121122		

$GF[29^2]$, $i^2 \equiv i + 26$, modulo 29, $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α , β	λ	α , β	λ	α , β
1	1, 0	20	22, 1	300	5
2	1,26	21	23,21	330	15
3	27,26	22	15,18	360	16
4	24, 6	23	4,13	390	19
5	1,15	24	17,17	420	28
6	16,26	25	5, 7	450	26
7	13,10	26	12,14	480	20
8	23,19	27	26,22	510	2
9	13,18	28	19, 9	540	6
10	2,19	29	28, 1	570	18
11	21,23	30	3	600	25
12	15,24	60	9	630	17
13	10,13	90	27	660	22
14	23,28	120	23	690	8
15	22,18	150	11	720	24
16	11,21	180	4	750	14
17	3,25	210	12	780	13
18	28,20	240	7	810	10
19	19, 3	270	21	840	1

SECOND TABLE.

λ	α , β	λ	α , β	λ	α , β
840	1	480	20	43	1,10
510	2	270	21	741	1,11
30	3	660	22	417	1,12
180	4	120	23	565	1,13
300	5	720	24	195	1,14
540	6	600	25	5	1,15
240	7	450	26	753	1,16
690	8	90	27	69	1,17
60	9	420	28	827	1,18
810	10	1	1, 0	522	1,19
150	11	234	1, 1	469	1,20
210	12	478	1, 2	728	1,21
780	13	67	1, 3	124	1,22
750	14	200	1, 4	706	1,23
330	15	734	1, 5	340	1,24
360	16	656	1, 6	683	1,25
630	17	532	1, 7	2	1,26
570	18	581	1, 8	486	1,27
390	19	438	1, 9	449	1,28

$GF[31^2]$, $i^2 \equiv i + 19$, modulo 31, $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α , β	λ	α , β	λ	α , β
1	1, 0	21	7, 3	320	25
2	1,19	22	10, 9	352	21
3	20,19	23	19, 4	384	4
4	8, 8	24	23,20	416	17
5	16,28	25	12, 3	448	18
6	13,25	26	15,11	480	30
7	7,30	27	26, 6	512	19
8	6, 9	28	1,29	544	11
9	15,21	29	30,19	576	8
10	5, 6	30	18,12	608	3
11	11, 2	31	30, 1	640	5
12	13,23	32	12	672	29
13	5,30	64	20	704	7
14	4, 2	96	23	736	22
15	6,14	128	28	768	16
16	20,21	160	26	800	6
17	10, 8	192	2	832	10
18	18, 4	224	24	864	27
19	22, 1	256	9	896	14
20	23,15	288	15	928	13
				960	1

SECOND TABLE.

λ	α , β	λ	α , β	λ	α , β
960	1	352	21	471	1,10
192	2	736	22	542	1,11
608	3	96	23	509	1,12
384	4	224	24	888	1,13
640	5	320	25	530	1,14
800	6	160	26	912	1,15
704	7	864	27	590	1,16
576	8	128	28	168	1,17
256	9	672	29	899	1,18
832	10	480	30	2	1,19
544	11	1	1, 0	681	1,20
32	12	388	1, 1	38	1,21
928	13	884	1, 2	263	1,22
896	14	427	1, 3	175	1,23
288	15	150	1, 4	243	1,24
768	16	827	1, 5	197	1,25
416	17	333	1, 6	330	1,26
448	18	145	1, 7	277	1,27
512	19	953	1, 8	44	1,28
64	20	698	1, 9	28	1,29
				511	1,30

INTEGRAL EQUATIONS.

An Introduction to the Study of Integral Equations. By MAXIME BÔCHER. Cambridge Tracts in Mathematics and Mathematical Physics, No. 10. Cambridge, The University Press, 1909. iv + 72 pages.

IN 1823 Abel proposed a generalization of the tautochrone problem whose solution involved the solution of an integral equation which has more recently been designated as an integral equation of the first kind, and in 1837 Liouville showed that the determination of a particular solution of a linear differential equation of the second order could be effected by solving an integral equation of a different type, called the integral equation of the second kind. The ripple of mathematical interest which had its origin in these investigations increased at first but slowly. Recently, however, stimulated by the researches of Volterra, Fredholm, and Hilbert in the period between 1896 and the present time, that which seemed at first only a ripple has grown into a formidable wave which bids fair to carry the integral equation theory into a place beside the most important of the mathematical disciplines. Notwithstanding the rapidly multiplying investigations in integral equations and the numerous applications of them which have been made, the sources of information concerning the theory have remained widely scattered and none too easily accessible to any but the specialist in the subject. It is with a hearty welcome, therefore, that the thoughtful mathematician will receive an introduction to the theory written by so clear a thinker and writer as the author of the book which is the subject of this review. As stated in the preface, the purpose of the author was to furnish the careful student with a firm foundation for further study, and at the same time so to display and arrange the principal theorems that one may with only a superficial reading obtain some idea of the subject. These objects seem to have been successfully attained. The book should furthermore be very useful as a text in an introductory course, especially if the instructor would content himself at first with the discussion of integral equations whose kernels are continuous or have discontinuities of the explicit forms which occur in the problem of Abel, treated in § 2, and other applications.

The emphasis which is placed on the historical development of the subject is an interesting feature of the book. After an introductory section in which some essential theorems concerning definite integrals are set down, the problems of Abel and Liouville, probably the earliest applications of integral equations, are discussed, and thereafter the reader's attention is constantly directed to the contributions which have been made by Volterra, Fredholm, Hilbert, Schmidt, Kneser, and other writers. In the papers of Hilbert and Schmidt the kernel of the integral equation is first assumed to be symmetric and continuous. Later they show that the theory for unsymmetric kernels can be regarded as an application of the theory for the symmetric case. Professor Bôcher, following the earlier writers, has inverted this arrangement, which seems more convenient since many of the principal results follow as easily for the unsymmetric as for the symmetric hypothesis. The author has also admitted from the start certain kinds of discontinuities in the kernels of his equations. This is perhaps disconcerting to the reader who wishes merely a survey of the theory, but the applications of equations with discontinuities are so frequent that one must feel that the admission is justified. It seems regrettable that more of the applications of the integral equation theory, for example Hilbert's unification of the theories of the expansion of an arbitrary function in terms of other functions and some of the applications to boundary value problems, could not have been introduced. The limited size of the book was evidently the preventive.

In commenting upon the theory as developed by Professor Bôcher, I shall not attempt to follow closely the order of his arrangement, but shall try to give an idea of the contents of the book as they have impressed themselves upon me. There are two kinds of integral equations which may be written in the forms

$$(1) \qquad f(x) = \int_a^b K(x, \xi)\, u(\xi) d\xi,$$

$$(2) \qquad u(x) = f(x) + \int_a^b K(x, \xi)\, u(\xi) d\xi,$$

where $K(x, \xi)$ and $f(x)$ are given functions, while $u(x)$ is to be determined. The problem proposed by Abel was to determine a curve $y = y(x)$ down which a heavy particle would fall

from a variable point (x, y) to the origin $(0, 0)$ in a time $T = f(x)/2g$, where $f(x)$ is an arbitrarily assigned function. The integral equation which gives the solution is one of the first kind, in which $K(x, y) \equiv 0$ for $y > x$. It has the form

$$f(x) = \int_0^x \frac{v'(\xi)d\xi}{\sqrt{(x-\xi)}},$$

where $v(x)$ is the length of arc measured from the origin and is to be determined. Liouville later made the determination of a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + [\rho^2 - \sigma(x)]y = 0,$$

where ρ is a constant, depend upon the solution of the integral equation

$$(3) \quad u(x) = \cos \rho(x - a) + \frac{1}{\rho}\int_a^x \sigma(\xi) \sin \rho(x - \xi)u(\xi)d\xi,$$

which is an equation of the second kind. In §§ 3, 5 Professor Bôcher exhibits the method which Liouville applied to the solution of these two equations, and applies the same method of successive substitutions to the general equation of the second kind.

The treatment devised by Volterra for equations of the second kind is both remarkable and elegant. It depends upon the notion of the iterated functions $K_i(x, y)$ defined by the formulas

$$K_1(x, y) = K(x, y), \quad K_i(x, y) = \int_a^b K(x, \xi)K_{i-1}(\xi, y)d\xi.$$

The series

$$(4) \qquad -k(x, y) = K_1 + K_2 + \cdots,$$

when it is uniformly convergent, determines uniquely a continuous function $k(x, y)$ which with $K(x, y)$ satisfies the equations

$$(5) \quad K(x, y) + k(x, y) = \int_a^b K(x, \xi)k(\xi, y)d\xi$$

$$= \int^b k(x, \xi)K(\xi, y)d\xi.$$

Any two functions K, k which have proper continuity properties and satisfy the last equations are said to be "reciprocal." By means of equations (5) it can be shown that the integral equation of the second kind has one and only one continuous solution, which is expressed by the formula

$$u(x) = f(x) - \int_a^b k(x, \xi) f(\xi) d\xi. \tag{6}$$

The solution by Volterra which has just been discussed depends for its validity upon the convergence of the series (4). Another method suggested by Volterra, but investigated by Fredholm, and later revised and extended by Hilbert, goes deeper into the meaning of the integral equations, explains the circumstances under which the reciprocal function $k(x, y)$ will or will not surely exist, and has besides an important application to integral equations involving an arbitrary parameter λ which will be mentioned later. Professor Bôcher shows in §7, following Fredholm, how one may regard the equation (2) as a limiting case for the system of equations

$$u_n(x_i) = f(x_i) + \sum_{j=1}^{n} K(x_i, x_j) u_n(x_j) \qquad (i = 1, 2, \cdots, n) \tag{7}$$

as n becomes infinite. Here $x_1, x_2, \cdots, x_n = b$ are supposed to divide the interval ab into n equal parts, and $u_n(x_1), u_n(x_2), \cdots, u(x_n)$ are the quantities to be determined. The determinant D_n of these equations goes over as n approaches infinity into an infinite series of integrals involving the kernel K, and if the values (x_μ, x_ν) have the limit (x, y) the corresponding cofactor $D_n(x_\mu, x_\nu)$ of D_n approaches a limiting value $D(x, y)$ which is also expansible into an infinite series. In § 8 the convergence of the series for D and the "adjoint" $D(x, y)$ is rigorously proved, and the important relations

$$-DK(x, y) + D(x, y) = \int_a^b K(x, \xi) D(\xi, y) d\xi$$

$$= \int_a^b D(x, \xi) K(\xi, y) d\xi$$

are derived. It follows at once that when $D \neq 0$ the function

$$k(x, y) = -\frac{D(x, y)}{D} \tag{8}$$

is reciprocal to $K(x, y)$, and equation (2) has a unique solution expressed by means of formula (6). On the other hand when $D = 0$ there will be no solution unless $f(x)$ satisfies

$$\int_a^b D(x, \xi) f(\xi) d\xi = 0,$$

a relation which is suggested by the condition which must be satisfied if equations (7) have a solution when $D_n = 0$.

The integral equation (3) which Liouville studied is of the form

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi) u(\xi) d\xi, \tag{9}$$

where $\lambda = 1/\rho$ and $K(x, y) \equiv 0$ for $y > x$. In § 9 Professor Bôcher begins the study of such equations. The determinant and adjoint function, as well as the reciprocal function, are here functions of the form $D(\lambda)$, $D(x, y, \lambda)$, $k(x, y, \lambda)$ containing the parameter λ which may take either real or complex values. The roots of $D(\lambda)$ are the "Eigenwerte" of Hilbert, or the "roots for the function $K(x, y)$." It is found in §§ 9, 10 that the necessary and sufficient condition that $K(x, y)$ have a reciprocal $k(x, y, \lambda)$ corresponding to a particular value of λ is that $D(\lambda) \neq 0$. If this condition is satisfied, equation (9) has a unique solution determined by equations (8) and (6).

The situation is somewhat different for the homogeneous equation

$$u(x) = \lambda \int_a^b K(x, \xi) u(\xi) d\xi, \tag{10}$$

as is explained in § 10. The unique solution of this equation when $D(\lambda) \neq 0$ is $u = 0$. On the other hand, for any root of $D(\lambda)$ the homogeneous equation has always an infinity of continuous solutions, called "principal solutions," which do not vanish identically. When $D = 0$ it follows from these results that the non-homogeneous equation (9) has either no continuous solution or else an infinite number found by adding to any particular solution of (9) the solutions of (10).

It was mentioned above that in the papers of Hilbert and Schmidt the theory of integral equations with unsymmetric kernel K has been made to depend upon that of equations in which the kernel is symmetric. In §§ 11, 12 Professor Bôcher

develops the theorems which relate especially to equations with symmetric kernels. For any such equation the determinant $D(\lambda)$ has at least one root, all the roots are necessarily real, and to any root of $D(\lambda)$ there corresponds only a finite number of linearly independent principal solutions of the homogeneous equation (10). A system $u_i(x)$ $(i = 1, 2, \cdots)$ of principal solutions belonging to roots of $D(\lambda)$ can be so chosen that any principal solution of equation (10) is expressible linearly and with constant coefficients in terms of a finite number of the functions $u_i(x)$, and furthermore so that

$$\int_a^b u_i^2(x)dx = 1, \quad \int_a^b u_i(x)u_j(x)dx = 0 \qquad (i \neq j).$$

A system of solutions having these properties is called "a complete normalized orthogonal system of characteristic functions for the kernel K." The trigonometric functions $\sin x$, $\sin 2x$, $\cdots$, are an example of such a system, in terms of which any function satisfying suitable restrictions can be expanded as an infinite series. Similar expansion theorems hold also for the system of characteristic functions belonging to any symmetric kernel. Professor Bôcher has restricted himself here, however, to the consideration of a single expansion, that for the kernel $K(x, y)$ itself, and to some of its applications.

The theory of the integral equations of the second kind having been developed, it is a comparatively simple matter to show, as Professor Bôcher does in § 13, that the solution of the integral equation

$$f(x) = \int_a^x K(x, \xi)\, u(\xi)d\xi, \tag{11}$$

which is one of the first kind with $K \equiv 0$ when $y > x$, are all solutions of the equation

$$f'(x) = K(x, x)\, u(x) + \int_a^x \frac{\partial K(x, \xi)}{\partial x}\, u(\xi)d\xi.$$

If $K(x, x)$ does not vanish in the interval ab, this is an equation of the second kind (2), and the problem of solving it is equivalent to the solution of the original equation (11) of the first kind. The case when $K(x, x)$ vanishes identically is treated, and an example illustrative of the case when $K(x, x)$ has a finite number of zeros is given. The section concludes with the study

of the more general equation of the first kind where $K(x, y)$ is assumed to have a discontinuity along a curve $y = \phi(x)$.

In the section just described the kernel was assumed to be finite. The earliest integral equation of the first kind, that of Abel, was however one in which the kernel became infinite along the line $x = y$. The concluding section of the book is devoted to equations of the type

$$f(x) = \int_a^x \frac{G(x, \xi)}{(x-\xi)^\lambda} u(\xi) d\xi,$$

which has a kernel with an infinite discontinuity including Abel's kernel as a special case for $G = 1$, and to a number of examples not falling under the previous theory. Especially interesting is the explanation of the relation of Fourier's integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \cos(x\xi) \cos(\xi\xi_1) f(\xi_1) d\xi_1 d\xi$$

to the theory of integral equations in which the limits are infinite.

G. A. BLISS.

SHORTER NOTICES.

Grundlagen der Analysis. Von MORITZ PASCH. Ausgearbeitet unter Mitwirkung von CLEMENS THAER. Leipzig, Teubner, 1908. 8vo. vi + 140 pp.

THIS book presents an admirable attempt to develop the concept of the real number in a more exact logical fashion. There is no attempt to reduce the assumptions to a categorical set, and even their consistency is not considered; but they are everywhere clearly stated, the theorems follow by ready deductions, and the large number of definitions would seem to be put in an unusually clear way, and one especially well adapted to the purpose of the general argument.

The book opens with a consideration of the relation of things to names, of the notions of precede and follow, and of methods of mathematical proofs. This is followed by a treatment of sets, sequences, and series, leading up to integers. By subjecting the integers to the four fundamental operations, fractions, including decimal fractions, and negative numbers are intro-

duced, and the laws of calculation are shown to be still valid in the enlarged set. The author then extends the concept of the set to non-enumerably infinite aggregates, which allows him to introduce irrational numbers in the usual Dedekind fashion. Powers with both bases and exponents, arbitrary real numbers, and logarithms of arbitrary positive real numbers to arbitrary positive real bases are then treated. The book proper concludes with an exposition of some of the principal theorems of combinations, and the binomial theorem for a positive real exponent. An appendix is added to the book, giving extracts from some of the author's previous writings.

On account of the abstract nature of the subject matter, the book is not suitable for the beginner, but it should appeal strongly to every teacher of mathematics. Notwithstanding the abstractness of the subject, the matter is attractively arranged, and may be perused with profit by one who possesses general maturity, even without an extensive knowledge of the technique of mathematics. It is unfortunate that books of this type are so inaccessible to English readers.

F. W. OWENS.

Eine konforme Abbildung als zweidimensionale Logarithmentafel zur Rechnung mit komplexen Zahlen. By Dr. F. BENNECKE, Professor at the Victoria-Gymnasium in Potsdam, 1907.

THE above is the title of the Festschrift by Dr. Bennecke at the three hundredth anniversary of the establishment of the Royal Joachimsthal Gymnasium at Berlin. In the early paragraphs is given a statement of the literature of the subject. While the author makes no pretension of presenting an exhaustive bibliography, nevertheless the citations are suggestive of the historical development and of the general interest in graphic methods, particularly as applied to operations with complex numbers. The purpose of the pamphlet is to establish a method by which the logarithms of complex numbers may be calculated with sufficient accuracy for practical purposes by means of graphs. This purpose is accomplished by a conformal representation upon the (X, Y)-plane of the two systems of curves given by $x = \text{constant}$, $y = \text{constant}$, where

$$W = X + iY = \log z = \log (x + iy).$$

It is shown that any curve of either of the two resulting systems

in the (X, Y)-plane is congruent with any other curve of the same system, and that the curves of the two systems are likewise congruent with one another. The relation between the curves makes it possible to construct all of the necessary graphs by the aid of a single regular curve. The construction of these graphs is furthermore simplified by showing that, to find the logarithm of any number to the base ten, it is sufficient to map on the W-plane only that portion of the z-plane which lies in the first quadrant and between the circles whose radii are 100 and 1000 respectively.

The rectangular form of the functional region makes it feasible to divide that region into nine subdivisions, thus giving a more convenient arrangement for evaluating a logarithm than by means of one large chart. From the ten charts which follow the general discussion both logarithms and anti-logarithms of complex numbers may be approximately determined. Upon the assumptions made as to the accuracy in locating points of intersections of two curves and in reading the charts, the maximum error in the absolute value of the logarithm will not exceed 0.0005. This accuracy corresponds therefore to a three place logarithmic table. Of course, as in ordinary logarithms, these errors may accumulate in the process of computation, so that the final error may be larger.

Aside from a theoretical interest in the work of Dr. Bennecke as an exercise in mapping, students of physics and others will find the charts of practical value wherever computations involving the logarithms of complex numbers are necessary.

E. J. TOWNSEND.

Elementary Algebra. By J. W. A. YOUNG and LAMBERT L. JACKSON. New York, Appleton and Co., 1908. ix + 438 pp.

THE guiding principle of the authors in this book is "a minimum of mathematical theory and a fuller recognition of the utility of the subject." This does not mean that the logical value of algebra has been ignored, but rather that the proofs given are put in a form which will appeal to and satisfy the mind of an average high school pupil; when in a few places this seemed impossible, the authors have fallen back upon simple assumptions rather than upon the introduction of subtle distinctions and arguments savoring of higher mathematical methods. For example, instead of explaining the process of

multiplying quadratic surds by means of the theory of exponents as is usually done, they introduce this topic by the statement, "In multiplying expressions containing square roots, make use of the relation $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$", and no proof of this relation is given.

The book is by no means radical, as it covers the usual topics of elementary algebra in about the conventional order, and presents material more than sufficient to meet present college entrance requirements. The method is inductive throughout. It may be characterized as modern, since the terms function and variable are used, graphs are included, determinant solution of simultaneous linear equations and methods of detached coefficients are given.

An important feature is the number of oral exercises scattered throughout the book. The reviewer believes oral work has been too much neglected in the last decade; numerous simple oral exercises often fix a principle better than complicated written exercises and besides give excellent drill in thinking and oral expression. Unfortunately there is no index of terms defined, but summaries of definitions, notions, and processes at the end of each chapter partially meet this need and have the advantage of fixing operations in the minds of the pupils. Graphs are used in treating the topics variation, simultaneous equations, and the quadratic. Graphs are not presented as selected topics from analytical geometry, but are used only where they bring insight into algebraic processes. However, the terms graph of a function and graph of an equation are used interchangeably where it seems desirable to the reviewer to distinguish carefully between graph of a function and locus of an equation. Equivalent equations are defined but no statement of operations which lead to equivalent equations is given. "A proportion is an equation between two numbers" (page 173) is doubtless a misprint, for proportion is correctly defined in another place (page 162).

This is one of the good elementary algebras recently published and deserves careful examination by teachers considering a change in algebra texts.

ERNEST B. LYTLE.

NOTES.

At the annual meeting of the London mathematical society held on November 11, 1909, the following officers were elected for the present year: President, W. Niven; secretaries, A. E. H. Love and J. H. Grace; treasurer, Sir J. Larmor; three vice-presidents and nine other members of the council. The following papers were read: By G. H. Hardy, "The ordinal relations of the terms of a convergent sequence," "The applications of Dirichlet's series to Borel's exponential method of summation," and "Theorems relating to summability and non-convergence of slowly oscillating series"; by W. Esson, "Notes on synthetic geometry"; by H. Bateman, "Kummer's quartic surface as a wave surface"; by H. S. Carslaw, "Green's function in a wedge and other problems in the conduction of heat"; by J. L. S. Hatton, "The envelope of a line cut harmonically by two conics"; by F. H. Jackson, "On a case of a q-hypergeometric series."

The American commissioners of the International commission on the teaching of mathematics earnestly request all to whom questionnaires have been addressed to send in their replies at the earliest possible moment. If the country is to prepare a set of reports that shall show the work that is being done, and if these reports are to be ready when those of other countries are presented, a great deal of hard work must be done this winter. Prompt replies therefore are necessary.

The annual meeting of the Federation of teachers of the mathematical and the natural sciences will be held at Boston, December 27–28, in affiliation with the American association for the advancement of science. Besides the reports of various administrative committees, that of the committee of fifteen on a syllabus in geometry will be presented and discussed; the committees on the college entrance problem and on the bibliography of science teaching will also present their reports.

During the year six associations have joined the federation, making the total membership fourteen. The annual election of officers will be held at the annual meeting, and standing committees will be appointed.

At the meeting of the Syracuse section of the Association of teachers of mathematics in the Middle States and Maryland

held at Syracuse December 28, 1909, the following papers were read: "Questions, answers, and per cents.," by C. F. WHEELOCK; "Algebraic number and the equation," by J. M. TAYLOR; "Shortening the course in arithmetic," by S. WILLIAMS.

AT the meeting of the Paris academy of sciences on November 7 the following prizes for contributions to geometry during the last academic year were announced: Francœur prize (1,000 fr.) to DR. E. LEMOINE; Bordin prize (3,000 fr.), divided between Professor G. BAGNERA, of the University of Palermo, and Professor M. DE FRANCHIS, of the University of Catania.

THE house of Underwood and Underwood, of London, announces a series of stereopticon views for use in teaching geometry of space. The first 25, prepared by E. LANGLEY, illustrate elementary theorems, while 23 others, by Professor A. G. GREENHILL, relate to spherical curves, particularly catenaries.

THE publishing house of Gauthier-Villars, Paris, announces that the following mathematical books are in the press, and will appear in a short time: O. BLUMENTHAL, "Principes de la théorie des fonctions entières d'ordre infini;" E. BOREL, "Leçons sur la théorie de la croissance;" C. RIQUIER, "Les systèmes d'équations aux dériveés partielles."

IN accordance with the resolution adopted by the fourth international congress of mathematicians, the president of the congress has appointed the following committee on the unification of the vectorial notation: Professors Abraham, Ball, Hadamard, Langevin, Lori, Marcolongo, Prandtl, Stekeloff, Whitehead, E. B. Wilson.

PROFESSORS F. KLEIN, of the University of Göttingen, Sir J. LARMOR, of Cambridge University, and H. POINCARÉ, of the University of Paris, have been elected honorary members of the Calcutta mathematical society.

DR. J. M. MILLER has been appointed acting head of the department of mathematics at the technical college of Glasgow.

MR. J. R. WILTON has been appointed assistant lecturer on mathematics at the University of Sheffield.

MR. G. H. HERRIOT has been appointed lecturer in mathematics at the school of mining, Kingston, Ontario.

PROFESSOR A. G. WEBSTER delivered at the University of

Illinois, November 29 to December 1, a series of public lectures on the following subjects: "Great physical problems of the past, present and future," "Sound and its measurement," "The gyroscope and its practical applications," "Classification of mathematical physics with reference to mechanics," "Waves, ether and relativity."

AT the University of Michigan Dr. J. W. BRADSHAW has been promoted to an assistant professorship of mathematics; Mr. W. V. N. GARRETSON has been granted a leave of absence to study at Yale; Mr. F. M. DRYZER, of the University of Iowa, and Mr. T. H. HILDEBRANDT, of the University of Chicago, have been appointed instructors in mathematics. Professor CARL RUNGE, of Göttingen, will deliver a series of lectures at the University in February.

AT the University of North Dakota Miss F. BALCH has been appointed instructor in mathematics.

MR. J. W. RUSSELL, former lecturer of mathematics in Balliol and St. John's Colleges, Oxford, died November 4 at the age of 81 years.

CATALOGUES of second hand mathematical books: Ernst Carlebach, Hauptstrasse 136, Heidelberg, catalogue no. 313, 1283 titles in mathematics and exact sciences.

NEW PUBLICATIONS.

(In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor, PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.)*

I. HIGHER MATHEMATICS.

ANDRÉ (D.). Des notations mathématiques. Enumération, choix et usage. Paris, Gauthier-Villars, 1909. 8vo. 18 + 502 pp. Fr. 16.00

BECKER (G. F.) and VAN ORSTRAND (C. E.). Hyperbolic Functions. Washington, Smithsonian Institution, 1909. 8vo. 51 + 321 pp. Cloth. $4.00.

BOREL (E.). Die Elemente der Mathematik. Vom Verfasser genehmigte deutsche Ausgabe besorgt von P. Stäckel. 2ter Band: Geometrie. Mit 403 Textfiguren. Leipzig, Teubner, 1909. 8vo. 12 + 324 pp. Cloth. M. 6.40

ENZYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. VI. 1ter Teil. A. Geodäsie und Geophysik. Redigiert von P. Furtwängler und E. Wiechert. 3tes Heft. Leipzig, Teubner, 1909. 8vo. Pp. 245–372. M. 4.00

FURTWÄNGLER (P.). See ENZYKLOPÄDIE.

GIOVANETTI (G.). Relazioni di condizione tra i coefficienti di una equazione cubica completa con radici particolari. Abbiategrasso, Bollini, 1909. 8vo. 11 pp.

GOLDMANN (F.). Ponceletsche Polygone bei Kreisen. (Diss.) Breslau, 1909. 8vo. 69 pp.

HOFMANN (J.). Nachweis der Richtigkeit des Fermatschen Satzes. Leipzig, 1909. 8vo. M. 1.20

KRIENELKE (K.). J. H. Lamberts Philosophie der Mathematik. Halle, 1909. Berlin, Mayer. 8vo. 101 pp. M. 2.40

LICHTENSTEIN (L.). Zur Theorie der gewöhnlichen Differentialgleichungen und der partiellen Differentialgleichungen zweiter Ordnung. Berlin, 1909. 8vo. 40 pp. M. 1.50

MARSHALL (W.). The asymptotic representation of the elliptic cylinder functions. (Diss.) Zurich, 1909. (Reprint from Am. Jour. of Math., vol. 31 (1909), pp. 311-336.)

MERCER (J.). Functions of positive and negative type, and their connection with the theory of integral equations. London, Dulau, 1909. 4to. 2s.

NEUENDORFF (R.). Ueber Kreispunktpolarkurven. (Diss.) Kiel, 1908. 8vo. 30 pp.

NEUMANN (F.). Eine Verallgemeinerung der Cylinderfunktionen. (Progr.) Halberstadt, 1909. 4to. 25 pp.

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte I: Calcolo differenziale. 3a edizione, riveduta. Milano, Hoepli, 1909. 16mo. 12 + 310 pp.

PRASAD (G.). A textbook of differential calculus. With numerous examples and answers. London, Longmans, 1909. 8vo. 5s.

REYE (T.). Die Geometrie der Lage. Vorträge. 3te Abteilung. 4te, umgearbeitete und vermehrte Auflage. Leipzig, Kröner, 1910. 8vo. 8 + 253 pp. M. 8.00

RIQUIER (C.). Les systèmes d'équations aux dérivées partielles. Paris, Gauthier-Villars, 1910. 8vo. 27 + 591 pp. Fr. 20.00

SAGERET (E.). Etude sur le théorème de Fermat. Paris, Dulac, 1909. 8vo. 38 pp.

SCHLAMP. Die Entstehung der Kegelschnitte nach Maclaurin und Grassmann. (Progr.) Darmstadt, 1908. 4to. 26 pp.

SCHMID (A.). Anwendung der Cauchy-Lipschitzschen Methode auf lineare partielle Differentialgleichungen. (Progr.) Schönberg, 1909. 8vo. 30 pp.

SCHRÖDER (R.). Das Dreieck und seine Berührungskreise. Gross-Lichterfelde, 1909. 8vo. 41 pp.

SIMON (M.). Geschichte der Mathematik im Altertum in Verbindung mit antiker Kulturgeschichte. Berlin, Cassirer, 1909. 8vo. 17 + 401 pp. M. 13.00

STÄCKEL (P.). See BOREL (E.).

STAMPER (A. W.). A history of the teaching of elementary geometry. (Diss.) New York, Columbia University Press, 1909. 8vo. 10 + 163 pp. Cloth. $1.50

STURM (R.). Die Lehre von den geometrischen Verwandtschaften. Band IV: Die nicht linearen und mehrdeutigen Verwandtschaften. Leipzig, Teubner, 1909. 8vo. 6 + 476 pp. Cloth. M. 18.00

VAN ORSTRAND (C. E.). See BECKER (G. F.).

WIDDER (W.). Untersuchungen über die allgemeinste lineare Substitution mit vorgegebener p^{ter} Potenz. (Diss.) Würzburg, 1909. 4to. 53 pp.

WIECHERT (E.). See ENZYKLOPÄDIE.

ZILLIG (J.). Ueber die infinitesimale Deformation der Minimalflächen. (Diss.) Würzburg, 1908. 8vo. 39 pp.

II. ELEMENTARY MATHEMATICS.

BARONI (E.). Algebra e trigonometria, ad uso dei licei. Vol. II (algebra ed aritmetica) per il secondo e terzo anno di liceo. Firenze, Bemporad, 1909. 16mo. 213 pp. L. 1.75

BERSANO (G. B.). See GUASCO (M.).

BOURLET (C.). Précis d'algèbre, contenant 573 exercices et problèmes. Classes de troisième B et première C et D. 5e édition, revue et complétée. Paris, Hachette, 1909. 16mo. 443 pp. Fr. 2.50

BRUNO (G. M.). Algebra y trigonometría con numerosos ejercicios. Paris, 1909. 12mo. 256 pp.

CAJORI (F.). A history of the logarithmic slide rule and allied instruments. London, Constable, 1909. 8vo. 6 + 144 pp. 4s. 6d.

C. (F. I.). Eléments d'algèbre, avec de nombreux exercices. Paris, Poussielgue, 1910. 12mo. 8 + 413 pp.

CLARKE (F. C.). See STARLING (S. G.).

CONANT (L. L.). Plane and spherical trigonometry. New York, American Book Co., 1909. 8vo. 222 pp. Cloth. $0.85
With 5-place logarithmic and trigonometric tables. $1.20

DAVISSON (S. C.). College algebra. New York, Macmillan, 1909. 12mo. 9 + 191 pp. $1.50

DIA (A. DI). Elementi di algebra, con numerosi esercizi, per uso delle scuole tecniche. Milano, Signorelli, 1909. 16mo. 78 pp. L. 1.00

EUCLIDES. The 13 books of Euclid's Elements. Translated from the text of Heiberg, with introduction and commentary, by T. L. Heath. 3 vols. Cambridge, 1908–09. 8vo. 439, 436, 554 pp. £2, 2s.

FAIFOFER (A.). Elementi di algebra, ad uso della prima classe liceale 2a edizione. Venezia, Sorteni, 1909. 16mo. 195 pp. L. 2.00

——. Elementi di geometria, ad uso degli instituti tecnici (1o biennio) e dei licei. 16a edizione. Venezia, Sorteni, 1909. 8vo. 534 pp. L. 4.00

——. Elementi di geometria, ad uso dei licei. Tredicesima edizione, ricavata dalla decimosesta edizione della stessa opera, destinata ai licei ed agl' istituti tecnici. Venezia, Sorteni, 1909. 16mo. 439 pp. L. 3.00

——. Trattato di geometria intuitiva, ad uso delle scuole tecniche e normali. 40a edizione. Venezia, Sorteni, 1909. 8vo. 165 pp. L. 2.00

GÉOMÉTRIE. Cours moyen. Paris, Poussielgue, 1909. 12mo. 191 pp.

GODFREY (C.) and SIDDONS (A. W.). Geometry for beginners. Cambridge, University Press, 1909. 8vo. 90 pp. 1s.

GUASCO (M.) e BERSANO (G. B.). Aritmetica, geometria e computisteria per le tre classi delle scuole complementari femminili. Torino, 1909. 8vo. 315 pp. L. 3.60

HALL (H. S.) and STEVENS (F. H.). A school geometry. Parts 3, 4 and 5. London, Macmillan, 1909. 8vo. 232 pp. 2s. 6d.

HAUCK (G.). Lehrbuch der Stereometrie. Auf Grund von F. Kommerell's Lehrbuch neubearbeitet und erweitert. 10te (umgearbeitete) Auflage. Herausgegeben von V. Kommerell. Tübingen, Laupp, 1909. 8vo. 16 + 170 pp. M. 2.00

HEATH (T. L.). See EUCLIDES.

KOMMERELL (V.). See HAUCK (G.).

LEPOIVRE (G.) et POIRSON (A.). Première partie du cours de géométrie théorique et pratique à l'usage des élèves des écoles pratiques d'industrie, des écoles professionnelles et des écoles primaires supérieures. Lille, Janny, 1909. 8vo. 164 pp.

MARTÍ ALPERA (F.). Las primeras lecciones de geometría. Grado elemental. Burgos, Santiago Rodríguez, 1909. 104 pp. P. 0.60

MARTINI ZUCCAGNI (A.). Aritmetica pratica e nozioni di geometria pel ginnasio inferiore. Vol. I, per la prima classe. Livorno, Giusti, 1909. 16mo. 147 pp. L. 1.40

——. Aritmetica razionale ed elementi di geometria euclidea (fondata sui postulati di Hilbert) pel ginnasio superiore. Vol. I, per la quarta classe. Livorno, Giusti, 1909. 8vo. 9 + 199 pp. L. 1.75

MATRICULATION mathematics papers. Being the papers in elementary mathematics set at the matriculation examination of the University of London from Jan. 1900 to Sept. 1909, with full solutions to the papers of Sept. 1909. London, Clive, 1909. 8vo. 136 pp. 1s. 6d.

MEHLER (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Bearbeitet von A. Schulte-Tigges. Ausgabe B. Oberstufe. 3ter Teil. Berlin, Reimer, 1909. 8vo. 7 + 84 pp. M. 1.50

MODEL answers to intermediate pure mathematics. Being the mathematical papers set at the London University intermediate examinations in arts and science from 1900 to 1909. London, Clive, 1909. 8vo. 196 pp. 2s. 6d.

MÜLLER (H. H.). See REINHARDT (W.).

NAMPON (G.). Cent questions de théorie (arithmétique, algèbre, géométrie) des examens du brevet supérieur, avec développements et solutions. Paris, Hachette, 1909. 16mo. 111 pp. Fr. 1.00

NOCIONES elementales de geometría aplicadas al dibujo lineal; por los hermanos de las escuelas cristianas, 14a edición. Paris, Procuraduría general, 1909. 12mo. 112 pp.

OTTO (J.). Ueber die Anwendung der Lehre von den Permutationen in der Algebra. (Progr.) Mährisch — Ostrau, 1909. 8vo. 22 pp.

POIRSON (A.). See LEPOIVRE (G.).

REINHARDT (W.) und MÜLLER (H. H.). Lehrbuch für den mathematischen Unterricht auf Grund der neuen Lehrpläne. In zwei Teilen im

Anschluss an die Aufgaben für den Rechenunterricht in höheren und mittleren Lehranstalten von J. C. Becker und K. Paul herausgegeben. Frankfurt a./M., Auffarth, 1909. 8vo.
1ter Teil. Arithmetik und Algebra. 7 + 120 pp. M. 1 60
2ter Teil. Geometrie. 7 + 136 pp. M. 1.80

ROPERT (J.). Cours d'algèbre élémentaire (deuxième degré) et propriétés usuelles des trois coniques et de l'hélice. Cours de troisième division, professé à l'école nationale d'arts et métiers de Lille. Angers, Lenormand, 1909. 8vo. 241 pp.

SIDDONS (A. W.). See GODFREY (C.).

STARLING (S. G.) and CLARKE (F. C.). Preliminary practical mathematics. London, Arnold, 1909. 8vo. 192 pp. 2s.

STEVENS (F. H.). See HALL (H. S.).

TRAVERSO (G.). Costruzioni geometriche, ad uso delle scuole secondarie. Parte I. 2a edizione. Savona, Prudente, 1909. 16mo. 81 pp.

WOODWARD (C. J.). ABC five-figure logarithms for general use. New edition. London, 1909. 8vo. 3s. 6d.

III. APPLIED MATHEMATICS.

BERGET (A.). See CHAPPUIS (J.).

BERNHARD (M.). Darstellende Geometrie mit Einschluss der Schattenkonstruktionen und der Perspektive. Als Leitfaden für den Unterricht an technischen Lehranstalten, Oberrealschulen und Realgymnasien, sowie zum Selbststudium herausgegeben. 3te, verbesserte und vermehrte Auflage. Stuttgart, Enderlen, 1909. 8vo. 8 + 294 pp. Cloth. M. 5.80

BORCHARDT (H.). Beiträge zur Kenntnis der stationären elektrischen Flächenströmung und ihres magnetischen Feldes. Nebst einer Anwendung auf die Variation der erdmagnetischen Vertikalintensität. (Diss.) Kiel, 1909. 8vo. 48 pp.

CAIN (W.). Theory of solid and braced elastic arches. 2d edition. New York, Van Nostrand, 1909. 16mo. 7 + 190 pp. Boards. $0.50

—— Theory of steel-concrete arches and of vaulted structures. 5th edition, revised. New York, Van Nostrand, 1909. 16mo. 215 pp. Boards. $0.50

CHAPPUIS (J.) et BERGET (A.). Leçons de physique générale. 2e édition, entièrement refondue. Vol. 3: Acoustique et optique. Paris, Gauthier-Villars, 1909. 8vo. 7 + 503 pp. Fr. 14.00

FRITSCH (H.). Die gegenseitige Massenanziehung bei Newton und bei seinen Nachfolgern. (Progr.) Königsberg, 1909. 8vo. 36 pp.

FUCHS (O.). Theoretische und kinematographische Untersuchung von Dampfhämmern mit selbsttätiger Schiebersteuerung. Berlin, Springer, 1909. 8vo. 20 pp. M. 1.20.

GROGAN (E. S.). The economic calculus and its application to tariff. Newcastle, 1909. 8vo. 184 pp. 2s.

GUILLAUME (C. E.). Initiation à la mécanique. Ouvrage étranger à tout programme, dédié aux amis de l'enfance. Paris, Hachette, 1909. 16mo. 16 + 214 pp. Fr. 2.00.

JECKE (R. H.). Beiträge zur Geometrie der Bewegung. Rostock, 1909. 8vo. 42 pp. M. 1.80.

LIMASSET (L.). Théorie rationnelle du mouvement mélodique des sons. Reims, Matot, 1909. 4to. 317 pp.

LOW (D. A.). Applied mechanics. Embracing strength and elasticity of materials. London, Longmans, 1909. 8vo. 560 pp. 7s. 6d.

MANEUVRIER (G.). Traité élémentaire de mécanique. Nouvelle édition, entièrement refondue. Paris, Hachette, 1909. 16mo. 394 pp. Fr. 4.00.

MANNES (H.). Die Berechnung von Rohrnetzen städtischer Wasserleitungen. München, Oldenbourg, 1909. 8vo. 59 pp. M. 1.60

MARCHESINI (G.). Elementi di calcolo attuariale, ad uso degli istituti tecnici, delle scuole medie di commercio e degli istituti di previdenza. Udine, Del Bianco, 1910. 8vo. 6 + 126 pp. L. 3.00

PEABODY (C. H.) Thermodynamics of the steam-engine and other heat-engines. 6th edition, rewritten. New York, Wiley, 1909. 7 + 543 pp. Cloth. $5.00

PEARSON (K.) and others. On a practical theory of elliptic and pseudo-elliptic arches, with special reference to the ideal masonry arch. London, Dulau, 1909. 4to. 4s.

RISSER (R.). Etude sur l'établissement des tables de mortalité de population. Mortalité professionnelle; mortalité dans le cas de l'invalidité. Paris, Dulac, 1909. 8vo. 112 pp.

SCHUSTER (A.). An introduction to the theory of optics. 2nd edition, revised. London, Arnold, 1909. 8vo. 368 pp. 15s.

SMITH (H. E.). Strength of material. An elementary study prepared for the use of midshipmen at the U. S. Naval Academy. 2nd edition, revised. New York, Wiley, 1909. 12mo. 9 + 170 pp. Cloth. $1.25

STEINMETZ (C. P.). Theoretical elements of electrical engineering. 3d edition, thoroughly revised and greatly corrected. New York, McGraw-Hill, 1909. 8vo. 9 + 455 pp. Cloth. $4.00

TAPLA (J.). Vademekum der darstellenden Geometrie. Für Schüler gewerblicher Lehranstalten, für Schüler und Absolventen des Gymnasiums sowie für Praktiker. 2te, durchgesehene Auflage. Wien, Fromme, 1909. 8vo. 12 + 182 pp. M. 5.00

TOLLE (M.). Die Regelung der Kraftmaschinen. Berechnung und Konstruktion der Schwungräder, des Massenausgleichs und der Kraftmaschinenregler in elementarer Behandlung. 2te, verbesserte und vermehrte Auflage. Berlin, Springer, 1909. 8vo. 11 + 699 pp. Cloth. M. 26.00

WELLNER (G.). Die Flugmaschinen. Theorie und Praxis. Berechnung der Drachenflieger und Schraubenflieger. Wien, Hartleben, 1910. 8vo. 8 + 152 pp. M. 10.00

WOLFF (O.). Folgerungen aus dem dritten Kepler'schen Gesetze. Eine Studie für Naturfreunde. Dux, 1909. Teplitz-Schönau, Becker. 8vo. 29 pp. M. 1.50